AUTOMORPHISMS OF NORMAL QUASI-CIRCULAR DOMAINS

Let $m_1, \dots, m_n \in \mathbb{N}$ be relatively prime. Recall that a domain $D \subset \mathbb{C}^n$ is said to be $(m_1, \dots, m_n)$-*circular* (shortly *quasi-circular*) if

$$(\lambda^{m_1} z_1, \dots, \lambda^{m_n} z_n) \in D, \quad \lambda \in \mathbb{C},\ |\lambda| = 1, \quad (z_1, \dots, z_n) \in D.$$

The n-tulpe $(m_1, \dots, m_n)$ is called the *weight* of the quasi-circular domain D. If $m_1 = \dots = m_n = 1$, the domain D is called *circular*.

A classical result of H. Cartan [1] states that a biholomorphic map between bounded circular domains containing the origin in $\mathbb{C}^n$ which fixes the origin is the restriction of a complex linear isomorphism. In 1970 W. Kaup [3] showed that every origin-preserving automorphism of a bounded quasi-circular domain is a polynomial mapping.

Thus, for quasi-circular domains, the following problem arises naturally (in what follows, all domains are assumed to be bounded and contained in $\mathbb{C}^n$, unless stated otherwise).

Problem 1. Let D be quasi-circular domain and suppose f is an origin-preserving automorphism of D. Describe how the weight of D and the degree of the polynomial mapping f are related.

A. Yamamori [5] studied the Problem 1 in the case $\deg f = 1$.

An $(m_1, \dots, m_n)$-circular domain D is called *normal*, if $m_j \geq 2$, $j = 1, \dots, n$, and $\gcd(m_j, m_k) = 1$ for any j, k such that $m_j \neq m_k$.

Theorem 1 (cf. [5])**.** *Let D, G be normal quasi-circular domains with $0 \in D$, $0 \in G$, and let $f : D \longrightarrow G$ be biholomorphic mapping with $f(0) = 0$. Then f is a linear isomorphism.*

Recall that a domain D is called a *minimal domain with a center at* $z_0 \in D$ if $\mathrm{Vol}(G) \geq \mathrm{Vol}(D)$ for any biholomorphic mapping $\varphi : D \longrightarrow G$ with $\det J(\varphi, z_0) = 1$, where

$$J(\varphi, z) := \left[\frac{\partial \varphi_j}{\partial z_k}(z)\right]_{j,k=1,\dots,n}$$

is the Jacobian matrix of $\varphi = (\varphi_1, \dots, \varphi_n)$ at $z = (z_1, \dots, z_n) \in D$. We shall use the following characterization of the minimality:

Proposition 2 (cf. [4])**.** *A domain D is a minimal domain with the center at z_0 iff $K_D(\cdot, z_0) \equiv c \neq 0$ on D.*

Example 3. The unit disc $\mathbb{D}$ is a minimal domain with center at the origin. Indeed, since

$$K_{\mathbb{D}}(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}, \quad z, w \in \mathbb{D},$$

we have $K_{\mathbb{D}}(\cdot, 0) = 1/\pi$.

Proposition 4 (cf. [5])**.** *If a domain D is quasi-circular and $0 \in D$ then it is a minimal domain with the center at the origin.*

For a domain D and $z = (z_1, \dots, z_n), w = (w_1, \dots, w_n) \in D$ such that the Bergman kernel $K_D(z, w) \neq 0$ we define an $n \times n$ matrix

$$T_D(z, w) := \left[\frac{\partial^2}{\partial \bar{w}_j \partial z_k} \log K_D(z, w)\right]_{j,k=1,\dots,n}.$$

$T_D(z, z)$ is a positive definite Hermitian matrix for all $z \in D$.

A domain D is called a *representative domain (in the sense of Lu Qi-Keng)* if there is a point $z_0 \in D$ such that $T_D(\cdot, z_0) = \text{const}$ on D. The point z_0 is called the *center of the representative domain D*.

Example 5. The unit disc $\mathbb{D}$ is a representative domain with center at the origin. Indeed, since

$$T_{\mathbb{D}}(z, w) = \left[\frac{2}{(1 - z\bar{w})^2}\right], \quad z, w \in \mathbb{D},$$

we have $T_{\mathbb{D}}(\cdot, 0) = [2]$.

Proposition 6 (cf. [5])**.** *Let D be normal quasi-circular domain with $0 \in D$. Then it is a representative domain with the center at the origin.*

Let D be a domain and let $p \in D$. Put

$$U_p^D := \{z \in D : K_D(z,p) \neq 0\}$$

and define a mapping $\sigma_p^D : U_p^D \longrightarrow \mathbb{C}^n$ by

$$\sigma_p^D(z) := T_D(p,p)^{-1/2} \operatorname{grad}_{\bar{w}} \log \frac{K_D(z,w)}{K_D(p,w)}\bigg|_{w=p}, \quad z \in U_p^D,$$

where $T_D(p,p)^{1/2}$ stands for the unique positive semidefinite square root of the matrix $T_D(p,p)$ and for anti-holomorphic function $f : D \longrightarrow \mathbb{C}$ we set

$$\operatorname{grad}_{\bar{w}} f(w) := {}^t\left(\frac{\partial f}{\partial \bar{w}_1}(w), \ldots, \frac{\partial f}{\partial \bar{w}_n}(w)\right).$$

The mapping σ_p^D is called the *Bergman mapping defined at* p. One may check that (cf. [2])

$$\sigma_p^D(p) = 0, \tag{1}$$

$$J(\sigma_p^D, z) = T_D(p,p)^{-1/2} T_D(z,p), \quad z \in U_p^D. \tag{2}$$

For a domain G and a biholomorphic mapping $\varphi : D \longrightarrow G$ we define an $n \times n$ matrix

$$L(\varphi, p) := T_G(\varphi(p), \varphi(p))^{-1/2}\, \overline{{}^tJ(\varphi,p)^{-1}}\, T_D(p,p)^{1/2}.$$

Proposition 7 (cf. [2])**.** *Let D, G be domains, $p \in D$, and let $f : D \longrightarrow G$ be a biholomorphic mapping. Then the diagram*

$$\begin{array}{ccc} U_p^D & \xrightarrow{f|_{U_p^D}} & U_{f(p)}^G \\ {\scriptstyle \sigma_p^D}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{f(p)}^G} \\ \mathbb{C}^n & \xrightarrow{L(f,p)} & \mathbb{C}^n \end{array}$$

commutes.

Proposition 8 (cf. [5])**.** *Assume that D, G are minimal representative domains with the center at the origin. Then any biholomorphic mapping $f : D \longrightarrow G$ with $f(0) = 0$ is linear.*

Proof of Theorem 1. By Propositions 4 and 6 we know that D and G are minimal representative domains with the centers at the origin. Then, by Proposition 8, we conclude that f is linear. □

Proof of Proposition 8. Since D, G are both minimal, from Proposition 2 we conclude that $U_0^D = D$ and $U_w^G = G$. Since D is representative, we have $T_D(\cdot, 0) = T_D(0,0)$ on D. Hence (2) implies that

$$J(\sigma_0^D, z) = T_D(0,0)^{1/2}, \quad z \in D.$$

Consequently, σ_0^D is an affine transformation. By (1) we have $\sigma_0^D(z) = T_D(0,0)^{1/2} z$, i.e. σ_0^D is linear. Similarly we show that σ_0^G is linear. Hence Proposition 7 implies that the diagram

$$\begin{array}{ccc} D & \xrightarrow{f} & G \\ {\scriptstyle T_D(0,0)^{1/2} = \sigma_0^D}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_0^G = T_G(0,0)^{1/2}} \\ \mathbb{C}^n & \xrightarrow{L(f,0)} & \mathbb{C}^n \end{array}$$

commutes. In particular,

$$f(z) = T_G(0,0)^{-1/2} L(f,0) T_D(0,0)^{1/2} z, \quad z \in D,$$

i.e. f is linear. □

References

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