ON CARATHÉODORY COMPLETENESS

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Our talk consists of two parts.

1. On non-compact versions of Edwards' theorem

Let X be a topological space and let $C(X)$ be the set of all continuous functions on X. A convex cone $S \subset C(X)$ is a subset such that $\alpha f + \beta g \in S$ for any $f, g \in S$ and any $\alpha, \beta \geq 0$. In future we assume that any convex cone contains also constant functions on X. We each convex cone S and a point $x \in X$ we associate two sets:

(1) $J_x^S(X)$ - the set of all *Jensen measures* with barycenter at x which consists of Borel probability measures μ with compact support such that $\psi(x) \leq \int \psi d\mu$ for any $\psi \in X$;
(2) $R_x^S(X)$ - the set of all *representing measures* with barycenter at x which consists of Borel probability measures μ with compact support such that $\psi(x) = \int \psi d\mu$ for any $\psi \in X$;

Note that $R_x^S(X) \subset J_x^S(X)$.
For any function $\varphi \in C(X)$ we consider its S-envelope as

$$E_\varphi^S(x) = \sup\{\psi(x) : \psi \in S, \psi \leq \varphi\}.$$

In 1965 Edwards proved the following result:

Theorem 1. *Let X be a compact topological space and let φ be a lower semicontinuous function on X. Then*

$$E_\varphi^S(x) = \min\Big\{ \int \varphi d\mu : \mu \in J_x^S(X)\Big\}.$$

In 2013 Gogus, Perkins, and Poletsky proved the following non-compact version of Edwards' theorem

Theorem 2. *Let X be a locally compact σ-compact Hausdorff space and let φ be a continuous function on X. Then $E_\varphi^S \equiv -\infty$ or*

$$E_\varphi^S(x) = \min\Big\{ \int \varphi d\mu : \mu \in J_x^S(X)\Big\}.$$

We say that a topological space X is of GPP-type[1] if for any positive linear functional $L : C(X) \to \mathbb{R}$ there exists a compact subset $K \subset X$ such that $L(\varphi) = 0$ whenever $\varphi \in C(X)$ and $\varphi|_K \equiv 0$.

Our main results in this part are the following

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[1]From the names Gogus, Perkins, and Poletsky.

Theorem 3. *Let X be a normal topological space of GPP-type and let φ be a continuous function on X. Then $E^S_\varphi \equiv -\infty$ or*

$$E^S_\varphi(x) = \min\Big\{ \int \varphi d\mu : \mu \in J^S_x(X)\Big\}.$$

Theorem 4. *Let $D \subset \mathbb{C}^n$ be a domain and let $\zeta \in \partial D$. Then $X = D \cup \{\zeta\}$ is of GPP-type.*

Remark 5. Actually with similar method one can show that for any domain $D \subset \mathbb{R}^n$ and any compact set $K \subset \partial D$ the set $X = D \cup K$ is of GPP-type. However, we do not need this in the futur, therefore we prove just the special case.

Proof of Theorem 3. Fix $x \in X$. Then we have a functional

$$E^S(X) : C(X) \ni \varphi \mapsto E^S_\varphi(X) \in [-\infty, +\infty).$$

Note that $E^S(X)$ is a positive and superlinear functional and, therefore, (see e.g. Gogus,Perkins and Poletsky)

$$E^S_\varphi(X) = \min\{L(\varphi) : L : C(X) \to \mathbb{R} \text{ linear}, L \geq E^S(X)\}.$$

Since X is of GPP-type for any $L : C(X) \to \mathbb{R}$ positive linear[2] there exists a compact set $K \subset X$ such that $L(\varphi) = 0$ whenever $\varphi = 0$ on K. From the Riesz represenation theorem there exists a Borel probability measure μ with support in K such that $L(\varphi) = \int \varphi d\mu$. □

Proof of Theorem 4. Take a sequence R_j, r_j such that $R_j > r_j > R_{j+1}$ and $R_j \to 0$ (e.g., $R_j = \frac{1}{3^j}$ and $r_j = \frac{2}{3^{j+1}}$). Consider functions $\chi_j \in C^\infty(\mathbb{R})$ such that $0 \leq \chi_j \leq 1$ having the following properties:

$$\chi_1(t) = \begin{cases} 1 & t \geq R_1 \\ 0 & t \leq r_1 \end{cases}$$

and for any $k \geq 2$

$$\chi_k(t) = \begin{cases} 1 - \sum_{j=1}^{k-1} \chi_j(t) & t \geq R_k \\ 0 & t \leq r_k \end{cases}.$$

Note that $\sum_{k=1}^\infty \chi_k(t) = 1$ for $t > 0$. Moreover, $\chi_k(t) = 0$ for $t \geq R_{k-1}$ and $t \leq r_k$.

Put $A_1 = X \setminus \overline{\mathbb{B}}(\zeta, r_1)$ and $A_k = \overline{\mathbb{B}}(\zeta, R_{k-1}) \setminus \mathbb{B}(\zeta, r_k)$, $k \geq 2$. Note that A_k, $k \geq 2$ are compact sets and that $\chi_k(\|x\|) = 0$ for $x \in \mathbb{C}^n \setminus A_k$.

Fix $k \geq 2$. For any $m \geq 1$ we consider compact sets

$$K_{km} = \{z \in A_k \cap X : \operatorname{dist}(z, \partial D) \geq \frac{1}{m}\}.$$

Note that $\cup_{m=1}^\infty K_{km} = A_k \cap D$. We claim that there exists an $m = m(k)$ such that $L(\varphi) = 0$ whenever $\varphi \in C(X)$, $\varphi \geq 0$, and $\varphi = 0$ on $(X \setminus A_k) \cup K_{km}$. Indeed, assume that for any $m \geq 1$ there exists a $\varphi_m \in C(X)$, $\varphi_m \geq 0$, $\varphi = 0$ on $(X \setminus A_k) \cup K_{km}$, and $L(\varphi_m) = 1$. Consider a function $\varphi = \sum_{m=1}^\infty \varphi_m$. Then $\varphi \in C(X)$ and $L(\varphi) = +\infty$. A contradiction.

Similarly, we show that there exists an $m = m(1)$ such that $L(\varphi) = 0$ whenever $\varphi \in C(X)$, $\varphi \geq 0$, and $\varphi = 0$ on $(X \setminus A_1) \cup K_{1m}$. Using the linearity of L, we can get rid of the condition $\varphi \geq 0$.

Put $K = \Big(\cup_{k=1}^\infty K_{km(k)}\Big) \cup \{\zeta\}$. Note that $K \subset X$ is a compact set. We want to show that $L(\varphi) = 0$ whenever $\varphi \in C(X)$, $\varphi \geq 0$, and $\varphi = 0$ on K. Fix $\varphi \in C(X)$ such that $\varphi \geq 0$ and

[2]Positivity follows from the inequality $L \geq E^S(X)$.

$\varphi = 0$ on K. Fix $\epsilon > 0$. Since $\varphi(\zeta) = 0$, there exists a neighborhood U of ζ such that $\varphi < \epsilon$ on U. Take k_0 sufficiently big such that $\overline{\mathbb{B}}(\zeta) \cap X \subset U$. Put $\widetilde{\chi}(t) = 1 - \sum_{k=1}^{k_0} \chi_k(t)$. Then

$$\varphi(x) = \sum_{k=1}^{k_0} \chi_k(\|x\|)\varphi(x) + \widetilde{\chi}(\|x\|)\varphi(x).$$

Hence, $L(\varphi) = L(\widetilde{\chi}\varphi) \leq \epsilon L(1)$. Since $\epsilon > 0$ was arbitrary we get $L(\varphi) = 0$. □

Corollary 6. *Let $D \subset \mathbb{C}^n$ be a domain and let $\zeta \in \partial D$. We put $X = D \cup \{\zeta\}$ and $S = H^\infty(D) \cap C(X)$. Then for any $\varphi \in C(X)$ we have $E_\varphi^S(\zeta) = \min\{\int \varphi d\mu : \mu \in R_\zeta^S(X)\}$.*

Moreover, if $R_\zeta^S(X) = \{\delta_\zeta\}$ then $E_\varphi^S(\zeta) = \varphi(\zeta)$.

2. On Carathéodory completeness

Before we state main results of this section, let us recall some notions and results from one-dimensional analysis.

Let $\mathcal{M}$ denotes the set of all positive probability measure in $\mathbb{C}$ with compact support and let $\mu \in \mathcal{M}$. We define its Newton potential as $M(\xi) = \int \frac{1}{|w-\xi|} d\mu(w)$. The following result is a corollary of Fubini's theorem

Lemma 7. *For any $\zeta \in \mathbb{C}$ we have*

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{\mathbb{D}(\zeta,r)} |w - \zeta| \cdot M(w) d\mathcal{L}(w) = \mu(\{\zeta\}).$$

As a Corollary we get

Corollary 8. *Assume that $\mu(\{\zeta\}) = 0$. Then for any $\epsilon > 0$ the set*

$$\Pi(\epsilon) = \{w \in \mathbb{C} : |w - \zeta| \cdot M(w) > \epsilon\}$$

has the property

$$\lim_{r \to 0} \frac{\mathcal{L}(\Pi(\epsilon) \cap \mathbb{D}(\zeta, r))}{\pi r^2} = 0.$$

For a set $X \subset \mathbb{C}^n$ we put $A(X) = H^\infty(\operatorname{int} X) \cap C(X)$.

The main result of this section is the following.

Theorem 9. *Let $D \subset \mathbb{C}^n$ be a domain. Consider the following conditions:*

(1) for any $\zeta \in \partial D$ there exist no a Borel probability measure μ with compact support in $D \cup \{\zeta\}$ such that $\mu \neq \delta_\zeta$ and

$$|f(\zeta)| \leq \int_{D \cup \{\zeta\}} |f(w)| d\mu(w) \quad \text{for any } f \in A(D \cup \{\zeta\}).$$

(2) for any $\zeta \in \partial D$ there exist no a Borel probability measure μ with compact support in $D \cup \{\zeta\}$ such that $\mu \neq \delta_\zeta$ and

$$f(\zeta) = \int_{D \cup \{\zeta\}} f(w) d\mu(w) \quad \text{for any } f \in A(D \cup \{\zeta\}).$$

(3) for any $\zeta \in \partial D$ there exists an $f \in A(D \cup \{\zeta\})$ such that $f(\zeta) = 1$ and $|f| < 1$ on D.

(4) D is c-finitely compact.

(5) D is c-complete.

Then (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5)*. Moreover, if* $n = 1$ *then* (5) $\implies$ (1) *and, therefore, all the above conditions are equivalent.*

Proof of Theorem 9. Note that the implications (1) $\implies$ (2) and (3) $\implies$ (4) $\implies$ (5) are immediate.

So, we have to prove that (2) $\implies$ (3). Actually it follows from Corollary 6 and Bishop's 1/3-2/3 technique of construction of a peak function.

Assume that $n = 1$. Let us prove (5) $\implies$ (1). Assume that there exists a positive probability measure μ such that $\mu(D) = 1$ and

$$|f(\zeta)| \leq \int_D |f| d\mu \quad \text{for any } f \in A(D \cup \{\zeta\}).$$

Fix $f \in A(D \cup \{\zeta\})$. Then there exists a sequence $f_n \in H^\infty(D)$ such that $\|f_n\|_D \leq 17\|f\|_D$, f_n extends to be analytic in a neighborhood of ζ and f_n converges uniformly to f on any set of type $D \setminus \mathbb{D}(\zeta, \epsilon)$, where $\epsilon > 0$.

For any $\eta \in D$ we put

$$g_n(z) = \frac{f_n(z) - f_n(\eta)}{z - \eta}.$$

Note that $g_n \in H^\infty(D \cup \{\zeta\})$. Then

$$|g_n(\zeta)| \leq \int_D |g_n(w)| d\mu(w) \leq 2\|f_n\|_\infty M(\eta) \leq 34\|f\|_\infty M(\eta)$$

and, therefore,

$$|f_n(\zeta) - f_n(\eta)| \leq |\zeta - \eta| \cdot 2\|f_n\|_\infty M(\eta) \leq |\zeta - \eta| \cdot 34\|f\|_\infty M(\eta).$$

For any $\eta_1, \eta_2 \in D$ we have

$$|f(\eta_1) - f(\eta_2)| \leq 34\|f\|_\infty \cdot \big(|\zeta - \eta_1| \cdot M(\eta_1) + |\zeta - \eta_2| \cdot M(\eta_2)\big).$$

Take a sequence $\{\eta_\nu\}$ such that $\eta_\nu \to \zeta$ and $|\zeta - \eta_\nu| \cdot M(\eta_\nu) \leq \frac{1}{2^\nu}$. Then $\{\eta_\nu\}$ is a c-Cauchy sequence. A contradiction. □

Corollary 10. *Let* $D \subset \mathbb{C}^n$ *be a domain. Assume that for any* $\zeta \in \partial D$ *there does not exist a Borel probability measure* μ *with compact support in* $D \cup \{\zeta\}$ *such that* $\mu(D) = 1$ *and*

$$|f(\zeta)| \leq \int_D |f(w)| d\mu(w) \quad \textit{for any } f \in A(D \cup \{\zeta\}).$$

Then D *is c-finitely compact.*

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