AUTOMORPHISMS OF NORMAL QUASI-CIRCULAR DOMAINS

Let $m_1, \dots, m_n \in \mathbb{N}$ be relatively prime. Recall that a domain $D \subset \mathbb{C}^n$ is said to be $(m_1, \dots, m_n)$-*circular* (shortly *quasi-circular*) if

$$(\lambda^{m_1} z_1, \dots, \lambda^{m_n} z_n) \in D, \quad \lambda \in \mathbb{C},\ |\lambda| = 1, \quad (z_1, \dots, z_n) \in D.$$

If $m_1 = \dots = m_n = 1$, the domain D is called *circular*.

An $(m_1, \dots, m_n)$-circular domain D is called *normal*, if $m_j \geq 2$, $j = 1, \dots, n$, and $\gcd(m_j, m_k) = 1$ for any j, k such that $m_j \neq m_k$.

Recall that a bounded domain $D \subset \mathbb{C}^n$ is called a *minimal domain with a center at* $z_0 \in D$ if $\mathrm{Vol}(G) \geq \mathrm{Vol}(D)$ for any biholomorphic mapping $\varphi : D \longrightarrow G$ with $\det J(\varphi, z_0) = 1$, where

$$J(\varphi, z) := \left[\frac{\partial \varphi_j}{\partial z_k}(z)\right]_{j,k=1,\dots,n}$$

is the Jacobian matrix of $\varphi = (\varphi_1, \dots, \varphi_n)$ at $z = (z_1, \dots, z_n) \in D$. We shall also use the following relative invariance of the Bergman kernel under the biholomorphic mapping

$$K_D(z, w) = \overline{\det J(\varphi, w)} K_G(\varphi(z), \varphi(w)) \det J(\varphi, z), \quad z, w \in D, \tag{1}$$

and the following characterization of the minimality:

Proposition 1 (cf. [2]). *A bounded domain D is a minimal domain with the center at z_0 iff $K_D(\cdot, z_0) \equiv c \neq 0$ on D.*

Proposition 2 (cf. [3]). *If a bounded domain D is quasi-circular and $0 \in D$ then it is a minimal domain with the center at the origin.*

Proof of Proposition 2. For $\theta \in \mathbb{R}$ define $f_\theta : D \longrightarrow D$ by the formula

$$f_\theta(z_1, \dots, z_n) := (e^{im_1\theta} z_1, \dots, e^{im_n\theta} z_n), \quad (z_1, \dots, z_n) \in D.$$

Observe that f_θ is an automorphism of D with $J(f_\theta, z) = \mathrm{diag}(e^{im_1\theta}, \dots, e^{im_n\theta})$. Formula (1) implies

$$K_D(z, 0) = K_D(f_\theta(z), 0), \quad z \in D,$$

whence, using Taylor expansion of $K_D(\cdot, 0)$, we get

$$\sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}_+^n} a_k z_1^{k_1} \dots z_n^{k_n} = \sum_{k=(k_1,\dots,k_n)\in\mathbb{Z}_+^n} e^{i(\sum_{j=1}^n m_j k_j)\theta} a_k z_1^{k_1} \dots z_n^{k_n}.$$

In particular,

$$a_k = e^{i(\sum_{j=1}^n m_j k_j)\theta} a_k, \quad \theta \in \mathbb{R},\ k \in \mathbb{Z}_+^n.$$

Since $\sum_{j=1}^n m_j k_j \neq 0$ except for $k_1 = \dots = k_n = 0$, we conclude that $a_k = 0$ for all $k \in (\mathbb{Z}_+^n)_*$, i.e. $K_D(\cdot, 0) = \text{const}$ on D. On the other hand, $K_D(0, 0) > 0$ i.e. $K_D(\cdot, 0) = \text{const} \neq 0$ on D. □

For a bounded domain $D \subset \mathbb{C}^n$ and $z = (z_1, \dots, z_n), w = (w_1, \dots, w_n) \in D$ such that the Bergman kernel $K_D(z, w) \neq 0$ we define an $n \times n$ matrix

$$T_D(z, w) := \left[\frac{\partial^2}{\partial \bar{w}_j \partial z_k} \log K_D(z, w)\right]_{j,k=1,\dots,n}.$$

$T_D(z, z)$ is a positive definite Hermitian matrix for all $z \in D$. Moreover, if $G \subset \mathbb{C}^n$ is a domain and $\varphi : D \longrightarrow G$ is a biholomorphic mapping we have

$$T_D(z, w) = \overline{{}^t J(\varphi, w)} T_G(\varphi(z), \varphi(w)) J(\varphi, z). \tag{2}$$

A bounded domain D is called a *representative domain (in the sense of Lu Qi-Keng)* if there is a point $z_0 \in D$ such that $T_D(\cdot, z_0) = \text{const}$ on D. The point z_0 is called the *center of the representative domain D*.

Proposition 3 (cf. [3]). *Let D be normal quasi-circular domain with $0 \in D$. Then it is a representative domain with the center at the origin.*

Proof of Proposition 3. For simplicity we shall write

$$K_{\bar{j},k}(z,w) := \frac{\partial^2}{\partial \bar{w}_j \partial z_k} \log K_D(z,w), \quad j,k = 1,\dots,n.$$

By (2) for any $\theta \in \mathbb{R}$ we have

$$K_{\bar{j},k}(z,0) = e^{i(m_k - m_j)\theta} K_{\bar{j},k}(f_\theta(z),0), \quad j,k = 1,\dots,n. \tag{3}$$

By a similar argument to the one used in the proof of Proposition 2, we know that $K_{\bar{j},k}(\cdot,0) = \text{const}$ on D whenever $m_j = m_k$. So fix j and k such that $m_j \neq m_k$. Without loss of generality we may assume that $m_j < m_k$.

First we prove that $K_{\bar{j},k}(\cdot,0) = \text{const}$ on D. Applying Taylor expansion of $K_{\bar{j},k}(\cdot,0)$ in (3), we get

$$\sum_{l=(l_1,\dots,l_n)\in\mathbb{Z}_+^n} a_l z_1^{l_1}\dots z_n^{l_n} = \sum_{l=(l_1,\dots,l_n)\in\mathbb{Z}_+^n} e^{i(m_k - m_j + \sum_{s=1}^n m_s l_s)\theta} a_l z_1^{l_1}\dots z_n^{l_n}.$$

In particular,

$$a_l = e^{i(m_k - m_j + \sum_{s=1}^n m_s l_s)\theta} a_l, \quad \theta \in \mathbb{R},\ l \in \mathbb{Z}_+^n.$$

Since $m_k - m_j + \sum_{s=1}^n m_s l_s \geq m_k - m_j > 0$ for all $(l_1,\dots,l_n) \in \mathbb{Z}_+^n$, we conclude that $a_l = 0$ for all $l \in \mathbb{Z}_+^n$, i.e. $K_{\bar{j},k}(\cdot,0) = 0$ on D.

Now we prove that $K_{\bar{k},j}(\cdot,0) = \text{const}$ on D. Again, applying Taylor expansion of $K_{\bar{k},j}(\cdot,0)$ in (3), we get

$$\sum_{l=(l_1,\dots,l_n)\in\mathbb{Z}_+^n} a_l z_1^{l_1}\dots z_n^{l_n} = \sum_{l=(l_1,\dots,l_n)\in\mathbb{Z}_+^n} e^{i(m_j - m_k + \sum_{s=1}^n m_s l_s)\theta} a_l z_1^{l_1}\dots z_n^{l_n}.$$

In particular,

$$a_l = e^{i(m_j - m_k + \sum_{s=1}^n m_s l_s)\theta} a_l, \quad \theta \in \mathbb{R},\ l \in \mathbb{Z}_+^n.$$

Since $c_{l,m} := m_j - m_k + \sum_{s=1}^n m_s l_s > 0$ for all $l = (l_1,\dots,l_n) \in \mathbb{Z}_+^n$ with $l_k \geq 1$, we conclude that $a_l = 0$ for all $l \in \mathbb{Z}_+^n$ with $l_k \geq 1$.

It remains to show that

$$c_{l,m} \neq 0 \quad \text{for } l = (l_1,\dots,l_n) \in (\mathbb{Z}_+^n)_* \text{ such that } l_k = 0. \tag{4}$$

Suppose (4) does not hold. Then

$$c_{l,m} = 0 \quad \text{for some } l = (l_1,\dots,l_n) \in (\mathbb{Z}_+^n)_* \text{ with } l_k = 0. \tag{5}$$

If $n = 2$ then the condition (5) has the form (without loss of generality we may assume that $j = 1$, $k = 2$)

$$m_1(l_1 + 1) = m_2, \qquad \text{for some } l_1 \in \mathbb{Z}_+,\ l_1 \geq 1,$$

which is impossible since, by normality, $\gcd(m_1, m_2) = 1$. Consequently, by (4), $a_l = 0$ for all $l \in (\mathbb{Z}_+^n)_*$ i.e. $K_{\bar{k},j}(\cdot,0) = \text{const}$ on D.

If $n \geq 3$, however, we cannot reason as above. Indeed, for $n = 3$, $m_1 = 2$, $m_2 = 3$, $m_3 = 5$ (observe that $(2,3,5)$-circular domain is normal), $j = 1$, $k = 3$, and $l = (0,1,0) \in (\mathbb{Z}_+^3)_*$ we have

$$c_{l,m} = m_1 - m_3 + m_2 = 2 - 5 + 3 = 0.$$

□

Let D be a domain and let $p \in D$. Put

$$U_p^D := \{z \in D : K_D(z,p) \neq 0\}$$

and define a mapping $\sigma_p^D : U_p^D \longrightarrow \mathbb{C}^n$ by

$$\sigma_p^D(z) := T_D(p,p)^{-1/2} \operatorname{grad}_{\bar{w}} \log \frac{K_D(z,w)}{K_D(p,w)}\bigg|_{w=p}, \quad z \in U_p^D,$$

where $T_D(p,p)^{1/2}$ stands for the unique positive semidefinite square root of the matrix $T_D(p,p)$ and for anti-holomorphic function $f : D \longrightarrow \mathbb{C}$ we set

$$\operatorname{grad}_{\bar{w}} f(w) := {}^t\left(\frac{\partial f}{\partial \bar{w}_1}(w),\dots,\frac{\partial f}{\partial \bar{w}_n}(w)\right).$$

The mapping σ_p^D is called the *Bergman mapping defined at* p. One may check that

$$\sigma_p^D(p) = 0, \tag{6}$$

$$J(\sigma_p^D, z) = T_D(p,p)^{-1/2} T_D(z,p), \quad z \in U_p^D. \tag{7}$$

Indeed, to see (7) note that

$$J(\sigma_p^D, z) = T_D(p,p)^{-1/2} J\left(\operatorname{grad}_{\bar{w}} \log \frac{K_D(\cdot, w)}{K_D(p,w)}\bigg|_{w=p}, z\right), \quad z \in U_p^D,$$

and

$$\frac{\partial}{\partial z_k}\left(\frac{\partial}{\partial \bar{w}_j} \log \frac{K_D(z,w)}{K_D(p,w)}\bigg|_{w=p}\right) = \frac{\partial^2}{\partial \bar{w}_j \partial z_k} \log \frac{K_D(z,w)}{K_D(p,w)}\bigg|_{w=p}, \quad z \in U_p^D,\ j,k = 1,\ldots,n.$$

For a domain G and a biholomorphic mapping $\varphi : D \longrightarrow G$ we define an $n \times n$ matrix

$$L(\varphi, p) := T_G(\varphi(p), \varphi(p))^{-1/2}\, \overline{{}^t J(\varphi,p)^{-1}}\, T_D(p,p)^{1/2}.$$

Proposition 4 (cf. [1])**.** *Let D, G be domains, $p \in D$, and let $f : D \longrightarrow G$ be a biholomorphic mapping. Then the diagram*

$$\begin{array}{ccc} U_p^D & \xrightarrow{f|_{U_p^D}} & U_{f(p)}^G \\ {\scriptstyle \sigma_p^D}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{f(p)}^G} \\ \mathbb{C}^n & \xrightarrow{L(f,p)} & \mathbb{C}^n \end{array}$$

commutes.

Proof of Proposition 3.3. Note that if $z \in U_p^D$, i.e. $K_D(z,p) \neq 0$ then, by (1), $K_G(f(z), f(p)) \neq 0$, i.e. $f(z) \in U_{f(p)}^G$. In particular, $f(U_p^D) \subset U_{f(p)}^G$.

For $z \in U_p^D$ by (1) we have

$$\frac{K_D(z,w)}{K_D(p,w)} = \frac{K_G(f(z), f(w)) \det J(f,z)}{K_G(f(p), f(w)) \det J(f,p)},$$

whence it follows that

$$\operatorname{grad}_{\bar{w}} \log \frac{K_D(z,w)}{K_D(p,w)}\bigg|_{w=p} = \operatorname{grad}_{\bar{w}} \log \frac{K_G(f(z), f(w))}{K_G(f(p), f(w))}\bigg|_{w=p}.$$

Using the change of variable $\xi := f(w)$ the right-hand side may be rewritten as

$$\overline{{}^t J(f,p)}\, \operatorname{grad}_{\bar{\xi}} \log \frac{K_G(f(z), \xi)}{K_G(f(p), \xi)}\bigg|_{\xi = f(p)}.$$ [1]

Consequently,

$$L(f,p)\sigma_p^D(z) = L(f,p) T_D(p,p)^{-1/2} \overline{{}^t J(f,p)}\, \operatorname{grad}_{\bar{\xi}} \log \frac{K_G(f(z), \xi)}{K_G(f(p), \xi)}\bigg|_{\xi = f(p)} = \sigma_{f(p)}^G(f(z)),$$

which ends the proof. □

References

[1] H. Ishi, C. Kai, *The representative domain of a homogeneous bounded domain*, Kyushu J. Math. **64** (2010), 35–47.
[2] M. Maschler, *Minimal domains and their Bergman kernel function*, Pacific J. Math. **6** (1956), 501–516.
[3] A. Yamamori, *Automorphisms of normal quasi-circular domains*, Bull. Sci. Math (2013), `http://dx.doi.org/10.1016/j.bulsci.2013.10.002`.

[1] Indeed, if $F(\lambda) := \log(K_G(f(z), \lambda)/K_G(f(p), \lambda))$, $\lambda \in G$, then

$$\overline{{}^t J(f,p)}\, \operatorname{grad}_{\bar{\xi}} F(\xi)|_{\xi=f(p)} = {}^t\left(\operatorname{grad}_{\bar{\xi}} F(\xi)|_{\xi=f(p)}\, \overline{J(f,p)}\right)$$
$$= {}^t\left(\left[\tfrac{\partial F}{\partial \bar{\xi}_1}(f(p)) \quad \ldots \quad \tfrac{\partial F}{\partial \bar{\xi}_n}(f(p))\right] \cdot \left[\tfrac{\partial \bar{f}_j}{\partial \bar{z}_k}(p)\right]_{j,k=1,\ldots,n}\right) = {}^t\left[\textstyle\sum_{j=1}^n \tfrac{\partial F}{\partial \bar{\xi}_j}(f(p)) \tfrac{\partial \bar{f}_j}{\partial \bar{z}_1}(p) \quad \ldots \quad \sum_{j=1}^n \tfrac{\partial F}{\partial \bar{\xi}_j}(f(p)) \tfrac{\partial \bar{f}_j}{\partial \bar{z}_n}(p)\right]$$
$$= {}^t\left[\tfrac{\partial (F \circ \bar{f})}{\partial \bar{z}_1}(p) \quad \ldots \quad \tfrac{\partial (F \circ \bar{f})}{\partial \bar{z}_n}(p)\right] = {}^t(\operatorname{grad}_{\bar{w}} F(f(w))|_{w=p}).$$