Connection between Bergman and Szegö kernels on the unit ball
based on S.G. Krantz *Geometric Analysis of the Bergman Kernel and Metric*, Chapter 1.9

Let Ω be a domain in $\mathbb{C}^n$. Let $A^2(\Omega) := \mathcal{O}(\Omega) \cap \mathcal{L}^2(\Omega)$.

Theorem 1. *For each $z \in \Omega$ the functional*

$$\Phi_z : A^2(\Omega) \ni f \mapsto f(z)$$

is a continuous linear functional on $A^2(\Omega)$.

From Riesz representation theorem for each $z \in \Omega$ there exists $K_z \in A^2(\Omega)$ such that

$$\Phi_z(f) = f(z) = \langle f, K_z \rangle = \int_\Omega f(\zeta) K_z(\zeta) \mathrm{d}V(\zeta).$$

Definition 1 (Bergman kernel)**.** *The function $K_\Omega(z,w) := \overline{K_z(w)}$, $z, w \in \Omega$, is called the* Bergman kernel *for Ω.*

It follows from the definition that for $f \in A^2(\Omega)$ the Bergman kernel has a reproducing property

$$f(z) = \int_\Omega K(z,\zeta) f(\zeta) \mathrm{d}V(\zeta), \;\; z \in \Omega.$$

Theorem 2. *If Ω is a bounded domain, then the mapping K where*

$$K(z) := \int_\Omega K(z,\zeta) f(\zeta) \mathrm{d}V(\zeta), \;\; f \in \mathcal{L}^2(\Omega)$$

is a projection of $\mathcal{L}^2(\Omega)$ onto $A^2(\Omega)$. It is called the Bergman projection.

Assume now that Ω is a bounded domain with $\mathcal{C}^2$ boundary. Let $H^2(\Omega) := \overline{\{f|_{\partial\Omega} : f \in \mathcal{O}(\Omega) \cap \mathcal{C}(\overline{\Omega})\}}$, where the closure is in the $\mathcal{L}^2(\partial\Omega)$.

Theorem 3. *For each $z \in \Omega$ the functional*

$$\Psi_z : H^2(\partial\Omega) \ni f \mapsto Pf(z),$$

where Pf denotes a Poisson integral of f, is continuous.

From Riesz representation theorem for each $z \in \Omega$ there exists $k_z \in H^2(\partial\Omega)$ such that

$$\Psi_z(f) = \langle f, k_z \rangle = \int_{\partial\Omega} f(\zeta) \overline{k_z(\zeta)} \mathrm{d}\sigma(\zeta).$$

Definition 2 (Szegö kernel)**.** *The function $S(z,\zeta) := \overline{k_z(\zeta)}$, $z \in \Omega$, $\zeta \in \partial\Omega$, is called the* Szegö kernel *for Ω.*

Theorem 4. *For $f \in H^2(\partial\Omega)$ the Szegö kernel has a reproducing property*

$$f(z) = \int_{\partial\Omega} S(z,\zeta) f(\zeta) \mathrm{d}\sigma(\zeta), \;\; z \in \Omega.$$

Theorem 5. *Let $\Omega = \mathbb{D}$, where $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$. Then:*

$$K(z,w) = \frac{1}{\pi} \frac{1}{(1 - z\overline{w})^2},$$

$$S(z,\zeta) = \frac{1}{2\pi} \frac{1}{1 - z\overline{\zeta}}.$$

Now we calculate the Szegö projection

$$Sf(z) = \int_{\partial\mathbb{D}} f(\zeta) S_{\mathbb{D}}(z,\zeta) \mathrm{d}\sigma(\zeta) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{1 - z\overline{\zeta}} \mathrm{d}\sigma(\zeta) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{1 - z\overline{\zeta}} \frac{\overline{\zeta}\mathrm{d}\zeta - \zeta \mathrm{d}\overline{\zeta}}{2i}$$

$$= \frac{1}{4\pi i} \left[\int_{\partial\mathbb{D}} f(\zeta) \frac{\overline{\zeta}}{1 - z\overline{\zeta}} \mathrm{d}\zeta - \int_{\partial\mathbb{D}} f(\zeta) \frac{\zeta}{1 - z\overline{\zeta}} \mathrm{d}\overline{\zeta} \right] = \frac{1}{4\pi i} \left[\int_{\mathbb{D}} \mathrm{d} \left(f(\zeta) \frac{\overline{\zeta}}{1 - z\overline{\zeta}} \mathrm{d}\zeta \right) - \int_{\mathbb{D}} \mathrm{d} \left(f(\zeta) \frac{\zeta}{1 - z\overline{\zeta}} \mathrm{d}\overline{\zeta} \right) \right]$$

$$= \frac{1}{4\pi i} \left[\int_{\mathbb{D}} \frac{\partial}{\partial\overline{\zeta}} \left(f(\zeta) \frac{\overline{\zeta}}{1 - z\overline{\zeta}} \right) \mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta - \int_{\mathbb{D}} \frac{\partial}{\partial\zeta} \left(f(\zeta) \frac{\zeta}{1 - z\overline{\zeta}} \right) \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta} \right]$$

$$= \frac{1}{4\pi i} \left[\int_{\mathbb{D}} \frac{\partial f}{\partial\overline{\zeta}} \frac{\overline{\zeta}}{1 - z\overline{\zeta}} + \frac{f(\zeta)}{(1 - z\overline{\zeta})^2} \mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta - \int_{\mathbb{D}} \frac{1}{1 - z\overline{\zeta}} \left(\frac{\partial f}{\partial\zeta} \zeta + f(\zeta) \right) \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta} \right]$$

$$= \frac{1}{4\pi i} \left[\int_{\mathbb{D}} \frac{\partial f}{\partial\overline{\zeta}} \frac{\overline{\zeta}}{1 - z\overline{\zeta}} \mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta + \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\overline{\zeta})^2} \mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta - \int_{\mathbb{D}} \frac{\partial f}{\partial\zeta} \frac{\zeta}{1 - z\overline{\zeta}} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta} - \int_{\mathbb{D}} \frac{f(\zeta)}{1 - z\overline{\zeta}} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta} \right]$$

$$= \frac{1}{4\pi i}\left[\int_{\mathbb{D}} \frac{\partial f}{\partial\overline{\zeta}} \frac{\overline{\zeta}}{1-z\overline{\zeta}} \mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta + \int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\overline{\zeta})^2}\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta - \int_{\mathbb{D}} \frac{\partial f}{\partial\zeta}\frac{\zeta}{1-z\overline{\zeta}}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} - \int_{\mathbb{D}} \frac{f(\zeta)-f(\zeta)z\overline{\zeta}}{(1-z\overline{\zeta})^2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right]$$

$$= \frac{1}{4\pi i}\left[\int_{\mathbb{D}} \frac{\partial f}{\partial\overline{\zeta}} \frac{\overline{\zeta}}{1-z\overline{\zeta}} \mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta + \int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\overline{\zeta})^2}\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta - \int_{\mathbb{D}} \frac{\partial f}{\partial\zeta}\frac{\zeta}{1-z\overline{\zeta}}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} + \int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\overline{\zeta})^2}\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta + \int_{\mathbb{D}} \frac{f(\zeta)z\overline{\zeta}}{(1-z\overline{\zeta})^2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right]$$

$$= \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\overline{\zeta})^2}\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta + \frac{1}{4\pi i}\left[\int_{\mathbb{D}} \frac{\partial f}{\partial\overline{\zeta}} \frac{\overline{\zeta}}{1-z\overline{\zeta}} \mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta - \int_{\mathbb{D}} \frac{\partial f}{\partial\zeta}\frac{\zeta}{1-z\overline{\zeta}}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} + \int_{\mathbb{D}} \frac{f(\zeta)z\overline{\zeta}}{(1-z\overline{\zeta})^2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right]$$

$$= \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\overline{\zeta})^2}\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i} + \frac{1}{4\pi i}\left[\int_{\mathbb{D}} \frac{\partial f}{\partial\overline{\zeta}} \frac{\overline{\zeta}}{1-z\overline{\zeta}} \mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta - \int_{\mathbb{D}} \frac{\partial f}{\partial\zeta}\frac{\zeta}{1-z\overline{\zeta}}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} + \int_{\mathbb{D}} \frac{f(\zeta)z\overline{\zeta}}{(1-z\overline{\zeta})^2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right]$$

$$= \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\overline{\zeta})^2}\mathrm{d}V(\zeta) + \frac{1}{4\pi i}\left[\int_{\mathbb{D}} \frac{\partial f}{\partial\overline{\zeta}} \frac{\overline{\zeta}}{1-z\overline{\zeta}} \mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta - \int_{\mathbb{D}} \frac{\partial f}{\partial\zeta}\frac{\zeta}{1-z\overline{\zeta}}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} + \int_{\mathbb{D}} \frac{f(\zeta)z\overline{\zeta}}{(1-z\overline{\zeta})^2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right]$$

$$= \int_{\mathbb{D}} f(\zeta)K_{\mathbb{D}}(z,\zeta)\mathrm{d}V(\zeta) + \frac{1}{4\pi i}\left[\int_{\mathbb{D}} \frac{\partial f}{\partial\overline{\zeta}} \frac{\overline{\zeta}}{1-z\overline{\zeta}} \mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta - \int_{\mathbb{D}} \frac{\partial f}{\partial\zeta}\frac{\zeta}{1-z\overline{\zeta}}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} + \int_{\mathbb{D}} \frac{f(\zeta)z\overline{\zeta}}{(1-z\overline{\zeta})^2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right],$$

where $\int_{\mathbb{D}} f(\zeta)K_{\mathbb{D}}(z,\zeta)\mathrm{d}V(\zeta)$ is the Bergman projection for the unit disc.

Now consider a monomial $f(z) = z^k\overline{z}^m$, $k,m\in\mathbb{N}$.

$$Sf(z) = Kf(z) + \frac{1}{4\pi i}\left[\int_{\mathbb{D}} \frac{\partial f}{\partial\overline{\zeta}} \frac{\overline{\zeta}}{1-z\overline{\zeta}} \mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta - \int_{\mathbb{D}} \frac{\partial f}{\partial\zeta}\frac{\zeta}{1-z\overline{\zeta}}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} + \int_{\mathbb{D}} \frac{f(\zeta)z\overline{\zeta}}{(1-z\overline{\zeta})^2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right]$$

$$= Kf(z) + \frac{1}{4\pi i}\left[z\int_{\mathbb{D}} \frac{\zeta^k\overline{\zeta}^{m+1}}{(1-z\overline{\zeta})^2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} - \int_{\mathbb{D}} k\frac{\zeta^k\overline{\zeta}^m}{1-z\overline{\zeta}}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} - \int_{\mathbb{D}} m\frac{\zeta^k\overline{\zeta}^m}{1-z\overline{\zeta}}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right]$$

$$= Kf(z) + \frac{1}{4\pi i}\left[zI_1 - (k+m)I_2\right]$$

We compute integrals I_1 and I_2 separately.

$$I_1 = \frac{1}{k+1}\int_{\partial\mathbb{D}} \frac{\zeta^{k-m}}{(1-z\overline{\zeta})^2}d\overline{\zeta} = \frac{1}{k+1}\int_{\partial\mathbb{D}} \frac{\overline{\zeta}^{m-k}}{(1-z\overline{\zeta})^2}d\overline{\zeta}$$

For $k \leqslant m$

$$\int_{\partial\mathbb{D}} \frac{\overline{\zeta}^{m-k}}{(1-z\overline{\zeta})^2}d\overline{\zeta} = 0$$

For $k > m$

$$\int_{\partial\mathbb{D}} \frac{\overline{\zeta}^{m-k}}{(1-z\overline{\zeta})^2}d\overline{\zeta} = -2\pi i(k-m)z^{k-m-1}$$

Thus

$$I_1 = \begin{cases} 0 & \text{for} \quad k\leqslant m \\ -2\pi i\frac{k-m}{k+1}z^{k-m-1} & \text{for} \quad k>m \end{cases}$$

Now we proceed to the second integral.

$$I_2 = \frac{1}{k+1}\int_{\partial\mathbb{D}} \frac{\zeta^{k+1-m}}{1-z\overline{\zeta}}d\overline{\zeta} = \frac{1}{k+1}\int_{\partial\mathbb{D}} \frac{\overline{\zeta}^{m-k-1}}{1-z\overline{\zeta}}d\overline{\zeta}$$

For $k < m$

$$\int_{\partial\mathbb{D}} \frac{\overline{\zeta}^{m-k-1}}{1-z\overline{\zeta}}d\overline{\zeta} = 0$$

For $k \geqslant m$

$$\int_{\partial\mathbb{D}} \frac{\overline{\zeta}^{m-k-1}}{1-z\overline{\zeta}}d\overline{\zeta} = -2\pi i z^{k-m}$$

Thus

$$I_2 = \begin{cases} 0 & \text{for} \quad k<m \\ -\frac{2\pi i}{k+1}z^{k-m} & \text{for} \quad k\geqslant m \end{cases}$$

Getting back to original equation we obtain: for $k<m$ $Sf(z) = Kf(z)$,
for $k=m$

$$Sf(z) = Kf(z) + \frac{1}{4\pi i}\left[-(k+m)\left(-2\pi i\frac{1}{k+1}z^{k-m}\right)\right] = Kf(z) + \frac{1}{2}\frac{k+m}{k+1}z^{k-m} = Kf(z) + \frac{m}{k+1}z^{k-m},$$

for $k > m$

$$\begin{aligned} Sf(z) &= Kf(z) + \frac{1}{4\pi i}\left[z\left(-2\pi i\frac{k-m}{k+1}z^{k-m-1}\right) - (k+m)\left(-2\pi i\frac{1}{k+1}z^{k-m}\right)\right] \\ &= Kf(z) + \frac{1}{2}\left[\frac{m-k}{k+1}z^{k-m} + \frac{k+m}{k+1}z^{k-m}\right] = Kf(z) + \frac{m}{k+1}z^{k-m}, \end{aligned}$$

and, finally

$$Sf(z) = \begin{cases} Kf(z) & \text{for} \quad k < m \\ Kf(z) + \frac{m}{k+1}z^{k-m} & \text{for} \quad k \geqslant m \end{cases}$$