The openness conjecture for plurisubharmonic functions (based on papers by Berndtsson and Demailly and Kollár)

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We would like to present the Berndtsson's proof of the openness conjecture (see [**2**]) posed by Demailly and Kollár) in [**4**].

Theorem 0.1. *Let u be a nonpositive plurisubharmonic function on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ such that*

$$\int_{\mathbb{B}} e^{-u} < \infty.$$

Then, there exist a number $\delta = \delta(n)$ and a number

$$p \geq 1 + \frac{\delta}{\int_{\mathbb{B}} e^{-u}}$$

such that

$$\int_{\frac{1}{2}\mathbb{B}} e^{-pu} < \infty.$$

The integrals are taken with respect to the Lebesgue measure. In order to give a sketch of proof of the above result, we need some notation. Let H_0 denote the space of square integrable holomorphic functions on $\mathbb{B}$. For u as in Theorem 0.1, let $H^2(\mathbb{B}, e^{-u})$ be the space of holomorphic functions on $\mathbb{B}$, square integrable with weight e^{-u}.

For any $s \geq 0$ we define a new norm on H_0 by

$$\|h\|_s^2 := \int_{\mathbb{B}} |h|^2 e^{-2u_s},$$

where $u_s := \max u + s, 0$. Observe that $\|\cdot\|_0$ is the standard unweighted L^2-norm. Moreover, $\{\|\cdot\|_s\}$ is a decreasing family of equivalent norms.

Let $E := H_0 \times \{\zeta \in \mathbb{C} : Re\zeta > 0\}$. This can be regarded as a trivial holomorphic vector bundle (of infinite rank) over $\{\zeta \in \mathbb{C} : Re\zeta > 0\}$. Moreover, the family $\|\cdot\|_{Re\zeta}$ gives a hermitian metric on E. Note that generally a hermitian metric on a vector bundle is assumed to be smooth. Our metric can be non-smooth (nevertheless, it is continuous, which will be important in the sequel - this follows from dominated convergence theorem) - this is a *singular* hermitian metric (see [**6**]). For such metrics we can introduce the notion of curvature. By definition, a vector bundle is positively curved, if its dual bundle is negatively curved. Furthermore, a vector bundle is negatively curved if the logarithm of the norm of any nonvanishing holomorphic section is plurisubharmonic (note that this definition is coherent with the one for finite rank vector bundles (Griffiths curvature) - see [**3**], [**6**]).

Using the main result of [**3**], we can deduce that E together with the hermition metric given by $\|\cdot\|_{Re\zeta}$ is positively curved:

Theorem 0.2. *Let E be as above. For $\zeta \in \{\zeta \in \mathbb{C} : Re\zeta > 0\}$ let v_ζ be a bounded plurisubharmonic function on $\mathbb{B}$. Define a hermitian metric on E by*

$$\|h\|_\zeta^2 = \int_{\mathbb{B}} |h|^2 e^{-v_\zeta}.$$

Finally, assume that the fuction $(\zeta, z) \mapsto v_\zeta(z)$ is plurisubharmonic. Then $(E, \|\cdot\|_\zeta)$ is positively curved (even in the sense of Nakano).

The key role in the proof of Teorem 0.1 is played by the following result (given in much more abstract setting).

Theorem 0.3. *Let H_0 be a separable Hilbert space equipped with a decreasing family of equivalent Hilbert norms $\|\cdot\|_s$ of positive curvature, defining new Hilbert spaces, H_s. Let*

$$H := H_0 \cap \{h : \|h\|^2 := \int_0^\infty e^s \|h\|_s^2 ds < \infty\}.$$

Then, for every $h \in H$, $\varepsilon > 0$, and $s > \frac{1}{\varepsilon}$ there exists an element $h_s \in H_0$ such that

$$\|h - h_s\|_0^2 \le 2\varepsilon \|h\|^2,$$

as well as

$$\|h_s\|_s^2 \le e^{-(1+\varepsilon)s} \|h\|_0^2.$$

The main ingredients of the proof of the above result are the following theorems:

Theorem 0.4 (Spectral theorem, **[7]**). *Let A be a bounded self-adjoint operator on a separable Hilbert space H. Then, there exist a finite measure space (X, μ), a bounded function F on X, and a unitary map $U : H \to L^2(X, d\mu)$ such that*

$$(UAU^{-1} f)(x) = F(x) f(x).$$

Theorem 0.5 (Maximum principle for positively curved hermitian metrics, **[1]**, see also **[5]**). *Let E be a holomorphic vector bundle over a smooth domain $D \subset\subset \mathbb{C}$ and let $h_)$, h_1 be two hermitian metrics on E that extend continuously to $\bar{D}$. Assume that the curvature of h_0 is flat and the curvature of h_1 is semi-positive. If $h_0 \le h_1$ on ∂D, then $h_0 \le h_1$ in D.*

The final step in an application of Theorem 0.3 for our vector bundle E and a function $h \equiv 1$, which belongs to H defined by Theorem 0.3. To see this, let ε_0 such that for any holomorphic function w with

$$\int_{\mathbb{B}} |w|^2 \le \varepsilon_0$$

there is $\sup_{\frac{1}{2}\mathbb{B}} |w| \le \frac{1}{10}$ (use Montel's theorem). Take $\varepsilon > 0$ so that

$$2\varepsilon \|h\|^2 = 2\varepsilon \int_{\mathbb{B}} e^{-u} = \varepsilon_0.$$

Then h_s given by Theorem 0.3 may be written in the form $1 - w$ (w as above) and one has

$$\int_{\frac{1}{2}\mathbb{B}} e^{-2u_s} \le 2\|h_s\|_s^2 \le 2e^{-s(1+\varepsilon)} \|h\|_0^2.$$

If we take $p = 1 + \frac{\varepsilon}{2}$, multiply the above inequality by e^{ps}, integrate from 0 to ∞, and make use of the following

Lemma 0.6. *For $f \in H^2(\mathbb{B}, e^{-u})$ and $p \in (0, 2)$ we have*

$$\int_{\mathbb{B}} |h|^2 e^{-pu} = A(p) \int_0^\infty e^{ps} \|h\|_s^2 ds + B(p) \|h\|_0^2.$$

we obtain

$$\int_{\frac{1}{2}\mathbb{B}} e^{-pu} < \infty.$$

Bibliography

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