Cantor Boundary Behavior of Analytic Functions

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**Proposition 1** Let $\Omega$ be a bounded simply connected domain. Let $f$ be a nonconstant analytic function in $\Omega$ and continuous on $\overline{\Omega}$. Suppose $\mathbb{C}_\infty \setminus f(\partial \Omega) = \bigcup_{j \geq 0} \mathbb{W}_j$ is the unique decomposition into components. Then:

(i) Each $\mathbb{W}_j$ is a simply connected domain.
(ii) $f^{-1}(f(\partial \Omega))$ is connected and each component of $\Omega \setminus f^{-1}(f(\partial \Omega))$ is a simply connected domain.

Let $n_f(w; K)$ denote the number of roots $z \in K$ for equation $f(z) = w$, counting according to multiplicity.

**Proposition 2** With the above assumption, suppose that $\mathbb{W}_j \cap f(\Omega) \neq \emptyset$. Let $f^{-1}(\mathbb{W}_j) = \bigcup_{k=1}^{q_j} O_j^k$ be the decomposition of the open set $f^{-1}(\mathbb{W}_j)$ into components. Then $1 \leq q_j < +\infty$; each $O_j^k$ is a simply connected component of $\Omega \setminus f^{-1}(f(\partial \Omega))$ and

$$f(O_j^k) = \mathbb{W}_j,$$

$$f(\partial O_j^k) = \partial \mathbb{W}_j.$$

Moreover, for each $w \in \mathbb{W}_j$, $n_f(w; O_j^k) \equiv n_{j,k}$ and $\sum_{k=1}^{q_j} n_{j,k} \equiv n_f(w; \Omega)$.

If, in addition $\partial \Omega$ is locally connected, then all the $\partial \mathbb{W}_j$ and $\partial O_j^k$ are locally connected.

**Proposition 3** With the above assumption and notation, $f'$ has $n_{j,k} - 1$ zeros in $O_j^k$.

The proof depends on the following lemma and the Riemann mapping theorem.

**Lemma 1** Let $f$ be analytic in $\mathbb{D}$ with $f(\mathbb{D}) = \mathbb{D}$. Suppose $n_f(w; \mathbb{D}) \equiv k$ for all $w \in \mathbb{D}$ for all $w \in \mathbb{D}$; then $f$ is a finite Blaschke product of degree $k$, and $f'(z)$ has $k - 1$ zeros in $\mathbb{D}$.

**Proposition 4** Let $f$ be a Blaschke product of degree $k$ and let $Z$ be a set of zeros of $f'$ in $\mathbb{D}^\circ$.

Suppose $f(Z) \subset L$ where $L$ is a Jordan curve in $\mathbb{D}_w$ except for an end point $\xi_0 \in \partial \mathbb{D}_w$. Let $G = \mathbb{D}_w \setminus L$ (it is simply connected), and let $f^{-1}(G) = \bigcup_{j=1}^{d} O_j$ be the connected component decomposition as in Proposition 2. Then $d = k$, and $f$ is univalent in $O_j$ with $f(O_j) = G$.

**Definition 1** Let $f$ be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. We say that $f$ has the Cantor boundary behaviour if $f^{-1}(\partial f(\mathbb{D}))$ and $\partial O \cap \partial \mathbb{D}$ are Cantor type sets in $\partial \mathbb{D}$ (whenever it is non-empty) where $O$ is any simply connected component of $\mathbb{D} \setminus f^{-1}(f(\partial \mathbb{D}))$ (as in Proposition 1).

**Lemma 2** Let $f$ be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. If there is a non-degenerated arc $J \subset \partial \mathbb{D}$ such that $f(J) \subset \partial f(\mathbb{D})$, then there exists a non-degenerated subarc $I \subset J$ and a bounded simply connected domain $D \subset \mathbb{D}$ such that $I \subset \partial D$, $\partial \mathbb{D}$ is locally connected, and $f$ is univalent in $D$.

[Sketch of proof]

**Lemma 3** Lemma 2 still holds if we replace the assumption $f(J) \subset \partial f(\mathbb{D})$ by $f(J) \subset \partial f(\mathbb{W})$ for some component $\mathbb{W}$ of $f(\mathbb{D}) \setminus f(\partial \mathbb{D})$. 1
Theorem 1 Let $f$ be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose the set of limit points of $Z = \{z \in \mathbb{D} : f'(z) = 0\}$ is $\partial \mathbb{D}$. Then $f$ has the Cantor boundary behavior.

Theorem 2 Let $f$ be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose, for any non-degenerated interval $I \subset [0, 2\pi]$, there exist $\kappa > 0$, $C > 0$, and $0 < r_0 < 1$ such that, for sufficiently small $\lambda > 0$,

$$\int_I |f'(re^{i\theta})|^\lambda d\theta \geq \frac{C}{(1-r)^\lambda \kappa}, \quad r_0 < r < 1.$$ 

Then $f$ has the Cantor boundary behavior.

Lemma 4 For $\theta_{k,m} := 2\pi mq^{-k}$ with $m = 0, \ldots, q^k - 1$, $k = 1, 2, \ldots$, there exist $C > 0$, $0 < \alpha < 1$, and $0 < \tau_k < \delta q^{-k}$ such that

$$\Re(e^{i\theta_{k,m}}f'(z)) \geq \frac{C}{(1-|z|)^\alpha}, \quad z \in S_\alpha(\theta_{k,m}, \tau_j) \setminus \{e^{i\theta_{k,m}}\}$$

In order to apply Theorem 2, it is more convenient to modify the integral mean growth condition to be discretized growth condition of $|f'|$.

Lemma 5 For $\theta_{k,m} := 2\pi mq^{-k}$ with $m = 0, \ldots, q^k - 1$, $k = 1, 2, \ldots$, suppose there exist $\kappa > 0$, $\delta > 0$, and $\eta \in (0, \frac{\pi}{2})$ such that

$$|f'(z)| \geq c(1-|z|)^{-\kappa}$$

for $z \in S_\eta(\theta_{k,m}, \frac{\delta}{2\pi})$ and $\frac{\delta}{2\pi} \leq 1 - |z| < \frac{\delta}{2\pi}$. Then the integral mean condition of Theorem 2 is satisfied.

Theorem 3 For $0 < \beta < 1$, $q \geq 2$ an integer, the complex Weierstrass function $f_{q, \beta}$ has Cantor boundary behavior.

Theorem 4 There exists a function $G$ such that, for any $z_{k,m}$,

$$F(z + z_{k,m}) = F(z_{k,m}) + G(z) z^{\alpha - 1} + z p_{k,m}(z), \quad 0 < \arg(z) < 2\pi,$$

where

(i) $G$ is continuous on $\mathbb{C} \setminus \{0\}$, analytic in $\Omega(\frac{\pi}{2})$ and $G(2z) = G(z)$ in $0 \leq \arg(z) < 2\pi$.

(ii) $p_{k,m}(z)$ is bounded continuous on $\mathbb{C}$, and analytic in $\Omega(\frac{\pi}{2}) \cup \{z : |z| < \frac{3}{2\pi + 1}\}$.

Proposition 5 There exists $C > 0$ such that

$$\max_{\text{dist}(z,K) \geq \epsilon} |F(z)| \leq Ct^{\alpha-2};$$

and the order is attained at the dyadic points of $\partial \Delta_0$, in the sense that there exists $0 < \eta < \frac{\pi}{2}$, $\delta > 0$ and $c > 0$ such that for any $z \in \Omega(\eta; 2^{-k} \delta)$,

$$|F'(z + z_{k,m})| \geq c|z|^\alpha.$$ 

Theorem 5 The Cauchy transform $F$ has the Cantor boundary behavior.

Theorem 6 The area of the Riemann region $F(\Delta_0)$ is finite, but it is infinite for $F(\mathbb{C} \setminus K)$.

Proposition 6 $\dim_H F(\partial \Delta_0) \leq (\alpha - 1)^{-1}(\approx 1.70951)$.

Conjecture 1 The box dimension and the Hausdorff dimension of $F(\partial \Delta_0)$ are $(\alpha - 1)^{-1}$.

Let $Gr(f; I) = \{(t, f(t)) : t \in I\}$ denote the graph of $f$ on an interval $I$.

Proposition 7 $\dim_B Gr(\Re(F); \partial \Delta_0)$ and $\dim_B Gr(\Im(F); \partial \Delta_0)$ are $3 - \alpha$. 
