Cantor Boundary Behavior of Analytic Functions

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Proposition 1 *Let Ω be a bounded simply connected domain. Let f be a nonconstant analytic function in Ω and continuous on $\overline{\Omega}$. Suppose $\mathbb{C}_\infty\backslash f(\partial\Omega) = \bigcup_{j\geq 0}\mathbb{W}_j$ is the unique decomposition into components. Then:*
(i) Each $\mathbb{W}_j$ is a simply connected domain.
(ii) $f^{-1}(f(\partial\Omega))$ is connected and each component of $\Omega\backslash f^{-1}(f(\partial\Omega))$ is a simply connected domain

Let $n_f(w;K)$ denote the number of roots $z\in K$ for equation $f(z)=w$, counting according to multiplicity.

Proposition 2 *With the above assumption, suppose that $\mathbb{W}_j\cap f(\Omega)\neq\emptyset$. Let $f^{-1}(\mathbb{W}_j) = \bigcup_{k=1}^{q_j}O_j^k$ be the decomposition of the open set $f^{-1}(\mathbb{W}_j)$ into components. Then $1\leq q_j<+\infty$; each O_j^k is a simply connected component of $\Omega\backslash f^{-1}(f(\partial\Omega))$ and*

$$f(O_j^k)=\mathbb{W}_j,\ f(\partial O_j^k)=\partial\mathbb{W}_j$$

Moreover, for each $w\in\mathbb{W}_j$, $n_f(w;O_j^k)\equiv n_{j,k}$ and $\sum_{k=1}^{q_j}n_{j,k}\equiv n_f(w,\Omega)$.
If, in addition $\partial\Omega$ is locally connected, then all the $\partial\mathbb{W}_j$ and ∂O_j^k are locally connected.

Proposition 3 *With the above assumption and notation, f' has $n_{j,k}-1$ zeros in O_j^k*

The proof depends on the following lemma and the Riemann mapping theorem.

Lemma 1 *Let f be analytic in $\mathbb{D}$ with $f(\mathbb{D})=\mathbb{D}$. Suppose $n_f(w;\mathbb{D})\equiv k$ for all $w\in\mathbb{D}$ for all $w\in\mathbb{D}$; then f is a finite Blaschke product of degree k, and $f'(z)$ has $k-1$ zeros in $\mathbb{D}$.*

Proposition 4 *Let f be a Blaschke product of degree k and let Z be a set of zeros of f' in $\mathbb{D}_z$. Suppose $f(Z)\subset L$ where L is a Jordan curve in $\mathbb{D}_w$ except for an end point $\xi_0\in\partial\mathbb{D}_w$. Let $G=\mathbb{D}_w\backslash L$ (it is simply connected), and let $f^{-1}(G)=\bigcup_{j=1}^{d}O_j$ be the connected component decomposition as in Proposition 2. Then $d=k$, and f is univalent in O_j with $f(O_j)=G$.*

Definition 1 *Let f be analytic in $\mathbb{D}$ and continouous on $\overline{\mathbb{D}}$. We say that f has the Cantor boundry behaviour if $f^{-1}(\partial f(\mathbb{D}))$ and $\partial O\cap\partial\mathbb{D}$ are Cantor type sets in $\partial\mathbb{D}$ (whenever it is non-empty) where O is any simply connected component of $\mathbb{D}\backslash f^{-1}(f(\partial\mathbb{D}))$ (as in Proposition 1).*

Lemma 2 *Let f be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. If there is a non-degenerated arc $J\subset\partial\mathbb{D}$ such that $f(J)\subset\partial f(\mathbb{D})$, then there exists a non-degenerated subarc $I\subset J$ and a bounded simply connected domain $D\subset\mathbb{D}$ such that $I\subset\partial D$, $\partial\mathbb{D}$ is locally connected, and f is univalent in D.*

[Sketch of proof]

Lemma 3 *Lemma 2 still holds if we replace the assumption $f(J)\subset\partial f(\mathbb{D})$ by $f(J)\subset\partial f(\mathbb{W})$ for some component $\mathbb{W}$ of $f(\mathbb{D})\backslash f(\partial\mathbb{D})$.*

Theorem 1 *Let f be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose the set of limit points of $Z = \{z \in \mathbb{D} : f'(z) = 0\}$ is $\partial\mathbb{D}$. Then f has the Cantor boundary behavior.*

Theorem 2 *Let f be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose, for any non-degenerated interval $I \subset [0, 2\pi]$, there exist $\kappa > 0$, $C > 0$, and $0 < r_0 < 1$ such that, for sufficiently small $\lambda > 0$,*

$$\int_I |f'(re^{i\theta})|^\lambda d\theta \geq \frac{C}{(1-r)^{\lambda\kappa}}, \quad r_0 < r < 1.$$

Then f has the Cantor boundary behavior.

Lemma 4 *For $\theta_{k,m} := 2\pi mq^{-k}$ with $m = 0, \ldots, q^k - 1$, $k = 1, 2, \ldots,$there exist $C > 0$, $0 < \alpha < 1$, and $0 < \tau_k < \delta q^{-k}$ such that*

$$\Re(e^{i\theta_{k,m}} f'(z)) \geq \frac{C}{(1-|z|)^\kappa}, \quad z \in S_\alpha(\theta_{k,m}, \tau_j)\backslash\{e^{i\theta_{k,m}}\}$$

In order to apply Theorem 2, it is more convenient to modify the integral mean growth condition to be discretized growth condition of $|f'|$.

Lemma 5 *For $\theta_{k,m} := 2\pi mq^{-k}$ with $m = 0, \ldots, q^k - 1$, $k = 1, 2, \ldots,$ suppose there exist $\kappa > 0$, $\delta > 0$, and $\eta \in (0, \frac{\pi}{2})$ such that*

$$|f'(z)| \geq c(1-|z|)^{-\kappa}$$

for $z \in S_\eta(\theta_{k,m}, \frac{\delta}{2^k})$ and $\frac{\delta}{2^{k+1}} \leq 1 - |z| < \frac{\delta}{2^k}$. Then the integral mean condition of Theorem 2 is satisfied.

Theorem 3 *For $0 < \beta < 1$, $q \geq 2$ an integer, the complex Weierstrass function $f_{q,\beta}$ has Cantor boundary behavior.*

Theorem 4 *There exists a function $\mathcal{G}$ such that, for any $z_{k,m}$,*

$$F(z + z_{k,m}) = F(z_{k,m}) + \mathcal{G}(z)z^{\alpha-1} + zp_{k,m}(z), \quad 0 < arg(z) < 2\pi,$$

where
(i) $\mathcal{G}$ is continuous on $\mathbb{C}\backslash\{0\}$, analytic in $\Omega(\frac{\pi}{2})$ and $\mathcal{G}(2z) = \mathcal{G}(z)$ in $0 \leq arg(z) < 2\pi$.
(ii) $p_{k,m}(z)$ is bounded continuous on $\mathbb{C}$, and analytic in $\Omega(\frac{\pi}{2}) \cup \{z : |z| < \frac{3}{2^{k+1}}\}$.

Proposition 5 *There exists $C > 0$ such that*

$$max_{dist(z,K)\geq t}|F'(z)| \leq Ct^{\alpha-2};$$

and the order is attained at the dyadic points of $\partial\Delta_0$, in the sense that there exists $0 < \eta < \frac{\pi}{2}$, $\delta > 0$ and $c > 0$ such that for any $z \in \Omega(\eta; 2^{-k}\delta)$,

$$|F'(z + z_{k,m})| \geq c|z|^{\alpha-2}.$$

Theorem 5 *The Cauchy transform F has the Cantor boundary behavior*

Theorem 6 *The area of the Riemann region $F(\Delta_0)$ is finite, but it is infinite for $F(\mathbb{C}\backslash K)$.*

Proposition 6 $dim_{\mathcal{H}} F(\partial\Delta_0) \leq (\alpha - 1)^{-1} (\approx 1.70951)$.

Conjecture 1 *The box dimension and the Hausdorff dimension of $F(\partial\Delta_0)$ are $(\alpha - 1)^{-1}$.*

Let $Gr(f; I) = \{(t, f(t)) : t \in I\}$ denote the graph of f on an interval I.

Proposition 7 *$dim_B Gr(\Re(F); \partial\Delta_0)$ and $dim_B Gr(\Im(F); \partial\Delta_0)$ are $3 - \alpha$.*