Cyclicity in Dirichlet-type spaces and extremal polynomials
(based on the paper by C. Bénéteau, A. Condori, C. Liaw, D. Seco, A. Sola)

For a number $\alpha \in \mathbb{R}$ we define the *Dirichlet-type space of order* α as

$$D_\alpha := \left\{ f = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{O}(\mathbb{D}) : \|f\|_\alpha < \infty \right\},$$

where

$$\|f\|_\alpha := \left(\sum_{k=0}^{\infty} |a_k|^2 (1+k)^\alpha \right)^{\frac{1}{2}}.$$

The space D_α is a Hilbert space with the inner product

$$\left\langle \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} b_k z^k \right\rangle_\alpha := \sum_{k=0}^{\infty} a_k \bar{b}_k (1+k)^\alpha.$$

Let $\varphi_\alpha(s) := s^{1-\alpha}$ for $\alpha < 1$, $s \geq 1$ and $\varphi_1(s) := \log s$ for $s \geq 1$. Denote by $\mathcal{P}_n$ the set of all polynomials in the variable z of degree not greater than n.

The main result of the considered paper is the following:

Theorem 1. *Let $\alpha \leq 1$, $f \in \mathcal{O}(\overline{\mathbb{D}}) \cap \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$. Then there exists a constant $C = C(\alpha, f) > 0$ such that for all sufficently big $n \in \mathbb{N}$ there holds*

$$(\operatorname{dist}_{D\alpha}(1, f \cdot \mathcal{P}_n))^2 \leq \frac{C}{\varphi_\alpha(n+1)}.$$

In particular, f is a cyclic vector for the operator $M_z : D_\alpha \ni g \mapsto z \cdot g \in D_\alpha$, i.e. the set $\{z^k f : k = 0, 1, \ldots\}$ is linearly dense in D_α.

Moreover, if $0 \in f(\partial\mathbb{D})$, then there exists a constant $C' = C'(\alpha, f) > 0$ such that for all sufficently big $n \in \mathbb{N}$ there holds

$$(\operatorname{dist}_{D\alpha}(1, f \cdot \mathcal{P}_n))^2 \geq \frac{C'}{\varphi_\alpha(n+1)}.$$