Notation

$||v||$ - standard Euclidean norm for $v \in \mathbb{C}^n$,

$\overline{\sigma}(M) = \sup\{|Mx| : |x| = 1\}$ = maximum eigenvalue of $(M^* M)^{1/2}$ - maximum singular value,

$\Delta_| := \{\text{diag}[\delta_1 I_{r_1}, ..., \delta_S I_{r_S}, \Delta_{S+1}, ..., \Delta_{S+F}], \}$,

where $\delta_i \in \mathbb{C}, \Delta_{S+j} \in \mathbb{C}^{m_j \times m_j}$

$\mathbb{B}_{\Delta_|} := \{\Delta \in \Delta_| : \overline{\sigma}(\Delta) \leq 1\}$

Definiton

For $M \in \mathbb{C}^{n\times n}$ we define

$$\mu_{\Delta_|}(M) := \frac{1}{\min\{\overline{\sigma}(\Delta) : \Delta \in \Delta_|, \det(I - M\Delta) = 0\}}$$

and we put $\mu_{\Delta_|}(M) := 0$ if for any $\Delta \in \Delta_|$ matrix $I - M\Delta$ is singular.
μ is continuous function.

Consider a complex matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

Definiton

We define a two blocks structure the following way:

$$\Delta_{|1} := \{\Delta_1 \textit{ is matrix } : M_{11}\Delta_1 \text{is squere}\}$$

,

$$\Delta_{|2} := \{\Delta_2 \textit{ is matrix } : M_{22}\Delta_1 \text{is squere}\}$$

For $\Delta_2 \in \Delta_{|2}$ consider the loop equations :

$$e = M_{11}d + M_{12}w,$$

$$z = M_{21}d + M_{22}w,$$

$$w = \Delta_2 z.$$

Definiton

Set of equations :

$$e = M_{11}d + M_{12}w,$$

$$z = M_{21}d + M_{22}w,$$

$$w = \Delta_2 z.$$

is called well posed if for any vector d, there exist unique vectors w, z and e satisfying the loop equations.

Observation

Equations are well posed if and only if $\det(I - M_{22}\Delta_2) \neq 0$

When the inverse does exist the vectors e and d satisfty:

$$e = P(M, \Delta_2)d$$

where

$$P(M, \Delta_2) := M_{11} + M_{12}\Delta_2(I - M_{22}\Delta_2)^{-1}M_{21}$$

Analogous formula describes $P(\Delta_1, M)$,

$$P(\Delta_1, M) := M_{22} + M_{21}\Delta_1(I - M_{11}\Delta_1)^{-1}M_{12}.$$

We can extend the definition P in the following way.
Suppose we have two complex matrix:

$$Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

with $Q_{22}M_{11}$ well defined and square. If $I - Q_{22}M_{11}$ is invertible then we define:

$$P(Q, M) := \begin{bmatrix} P(Q, M_{11}) & Q_{12}(I - M_{11}Q_{22})^{-1}M_{12} \\ M_{21}(I - Q_{22}M_{11})^{-1}Q_{21} & P(Q_{22}, M) \end{bmatrix}$$

Next we only consider equation:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P(Q, M)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Definiton

We define:

$$\mathbb{B}_i := \{\Delta_i \in \Delta_{|i} : \overline{\sigma}(\Delta_i) \leq 1\}$$

and structure $\Delta_|$ *as*

$$\Delta_| := \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta_1 \in \Delta_{|1}, \Delta_2 \in \Delta_{|2} \right\}$$

and

$$\mu_i := \mu_{\Delta_{|i}} \text{ for i} = 1,2$$

Theorem

The linear fractional transformation $P(M, \Delta_2)$ *is well possed for all* $\Delta_2 \in \mathbb{B}_2$ *if and only if* $\mu_2(M_{22}) < 1$

Theorem

(Main Loop Theorem)

$$\mu_\Delta(M) < 1 \Leftrightarrow \begin{cases} \mu_2(M_{22}) < 1, \\ \max_{\Delta_2 \in \mathcal{B}_2} \mu_1(P(M, \Delta_2)) < 1 \end{cases}$$

Proof.

Of course $\mu_\Delta(M) < 1$ implies that $\mu_2(M_{22}) < 1$ Let $\Delta_i \in \Delta_{|i}$ be such that $\overline{\sigma}(\Delta_i) \leq 1$, and define

$$\Delta = \mathrm{diag}[\Delta_1, \Delta_2]$$

. We see that

$$\det(I - M\Delta) = \det \begin{bmatrix} I - M_{11}\Delta_1 & -M_{12}\Delta_2 \\ -M_{21}\Delta_1 & I - M_{22}\Delta_2 \end{bmatrix}.$$

Because $I - M_{22}\Delta_2$ is invertible, hence

$$\det(I - M\Delta) = \det(I - M_{22}\Delta_2) \cdot \det(I - M_{11}\Delta_1 - M_{12}\Delta_2(I - M_{22}\Delta_2)^{-1}M_{21}\Delta_1).$$

Proof.

and therefore

$$\det(I - M\Delta) = \det(I - M_{22}\Delta_2)\det(I - P(M, \Delta_2)\Delta_1).$$

□

Example

Let $\Delta_{|1} := \{\delta_1 I_n : \delta_1 \in \mathbb{C}\}$, $\Delta_{|2} := \mathbb{C}^{m\times m}$. Recall that $\mu_1(A) = \varrho(A)$, $\mu_2 = \overline{\sigma}(D)$. Let A, B, C and D be given. Consider the state space model of a discrete time

$$x_{k+1} = Ax_k + Bu_k,$$

$$y_k = Cx_k + Du_k$$

and define

$$M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Example

The following conditions are equivalent:

- $\rho(A) < 1$ *and*

$$\max_{\delta_1 \in \mathbb{C}, |\delta_1| \leq 1} (D + C\delta_1(I - A\delta_1)^{-1}B) < 1$$

- $\overline{\sigma}(D) < 1$ *and*

$$\max_{\Delta_2 \in \mathbb{C}^{m \times m}, \overline{\sigma}(\Delta_2) \leq 1} \rho(A + B\Delta_2(I - D\Delta_2)^{-1}C < 1$$

- $\mu_{\Delta_|}(M) < 1$

Let $\Delta_{|1}$ and $\Delta_{|2}$ be two given structures. Define

$$\Delta_{|} := \{\text{diag}[\Delta_1, \Delta_2] : \Delta_i \in \Delta_{|i}\}$$

$$\mathbb{D}_i := \{\text{diag}[D_1, D_2, ..., D_S, d_{S+1} I_{m_1}, ..., d_{S+F} I_{m_F}] :$$
$$D_i = D_i^* > 0, d_{S+j} \in \mathbb{R}, d_{S+j} > 0\} \subset \Delta_{|i}$$
$$\mathbb{D} := \{\text{diag}[D_1, D_2] : D_i \in \mathbb{D}_i\}$$

Theorem

(Redheffer, 1959, 1960) Let M be

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

Suppose there is a $D \in \mathbb{D}$ such that $\overline{\sigma}(D^{1/2} M D^{-1/2}) < 1$. Then there exists a $D_1 \in \mathbb{D}_1$ such that

$$\max_{\Delta_2 \in \mathbb{B}_2} \overline{\sigma}(D_1^{1/2} P(M, \Delta_2) D_1^{-1/2}) < 1.$$

Proof.

Let D_1 and D_2 be the separate parts of the $D \in \mathbb{D}$ such that $\overline{\sigma}(D^{1/2} M D^{-1/2}) < 1$. We see that $\mu_2(M_{22}) < 1$ so for any $\Delta_2 \in \mathbb{B}_2$ the two LFT's are well possed (?). By assumption for $d \neq 0$ we find unique e, w, z such that

$$||z||^2 + ||e||^2 < ||w||^2 + ||d||^2$$

and since $\overline{\sigma}(\Delta_2) \leq 1$

$$||w||^2 \leq ||z||^2.$$

We get

$$||e||^2 < ||d||^2$$

□

BREAK

Recall that:
If $M \in \mathbb{C}^{n \times n}$ then we have sigular value decomposition

$$M = \sigma_1 UV^* + U_2 \Sigma_2 V_2^*$$

.
For U and V compatibly with $\Delta_|$

$$U = \begin{bmatrix} A_1 \\ . \\ . \\ A_S \\ E_1 \\ . \\ . \\ E_F \end{bmatrix} \qquad V = \begin{bmatrix} B_1 \\ . \\ . \\ B_S \\ H_1 \\ . \\ . \\ H_F \end{bmatrix}$$

we define

$$P_i^\eta := A_i \eta\eta^* A_i^* - B_i \eta\eta^* B_i^\eta,$$
$$p_{S+j}^\eta := \eta^*(E_j^* E_j - H_j^* H_j)\eta$$

and

$$\nabla_M := \{\text{diag}[P_1^\eta, ..., P_S^\eta, p_{S+1}^\eta I_{m_1}, ..., p_{S+F-1}^\eta I_{m_{F-1}}, O_{m_F}] : \eta \in \mathbb{C}^r, ||\eta|| = 1\}$$

We have known

Theorem

$$\inf_{D \in \mathbb{D}} \overline{\sigma}(D^{1/2} M D^{-1/2}) = \overline{\sigma}(M) \quad \text{iff} \quad 0 \in conv(\nabla_M)$$

Theorem

The following statements are equivalent

(a) $0 \in \nabla_M$

(b) *There exists* $\eta \in \mathbb{C}^r$, $||\eta|| = 1$ *and* $Q \in \mathbb{Q} := \{A^*A = I\}$ *with* $QU\eta = V\eta$

(c) $\overline{\sigma}(M) = \mu_\Delta(M)$

Definiton

Consider structure $\Delta_|$ *has the following property :*

if $W \in \mathbb{C}^{n \times n}$ *and* $0 \in conv(\nabla_W)$ *then* $0 \in \nabla_W$.

In this case we say $\Delta_|$ *is* μ*-simple.*

Theorem

Suppose the block structure Δ is μ-simple. Then for every $M \in \mathbb{C}^{n\times n}$ we have

$$\mu_\Delta(M) = \inf_{D\in\mathbb{D}} \overline{\sigma}(D^{1/2} M D^{-1/2})$$

Theorem

(Fact from Functional Analysis)

$$\rho(M) = \inf_{D\in\mathbb{C}^{n\times n}, D=D^*>0} \overline{\sigma}(D^{1/2} M^{-1/2})$$

In this section we will answer the question:
when we have

$$\mu_\Delta(M) = \inf_{D\in\mathbb{D}} \overline{\sigma}(D^{1/2} M D^{-1/2})?$$

Case $S = 0, F = 1$ is trivial.
In case $S = 1, F = 0$ we have

$$\rho(M) = \inf_{D \in \mathbb{C}^{n \times n}, D = D^* > 0} \overline{\sigma}(D^{1/2} M^{-1/2})$$

In this situation we have

$$\nabla = \{\eta^*(E^*E - F^*F)\eta : \eta \in \mathbb{C}^r, ||\eta|| = 1\}$$

Since $E^*E - F^*F$ is Hermitian, ∇ is a closed interval in the real line.

In this case we have

$$\nabla = \{A\eta\eta^* A^* - B\eta\eta^* B^* : \eta \in \mathbb{C}^r, ||\eta|| = 1\}$$

for some $r > 0$ and $A, B \in \mathbb{C}^{r_1 \times r}$.
Of course we see that ∇ is not convex ($A = I, B = 0$)

Theorem

Let ∇ be defined as

$$\nabla = \{A\eta\eta^* A^* - B\eta\eta^* B^* : \eta \in \mathbb{C}^r, ||\eta|| = 1\}$$

for some $r > 0$ and $A, B \in \mathbb{C}^{r_1 \times r}$.

If $0 \in \text{conv}(\nabla)$, then $0 \in \nabla$

Proof.

Suppose that $0 \in \text{conv}(\nabla)$. Then for some integer p. There exists $\alpha \in [0,1]$ with $\sum_{i=1}^{p}$ and vectors $\eta_i \in \mathbb{C}^r$ with $||\eta_i||$ such that

$$\sum_{i=1}^{p} \alpha_i (A\eta_i\eta_i^* A^* - B\eta_i\eta_i^* B^*) = 0$$

. We see that

$$A(\sum_{i=1}^{p} \alpha_i\eta_i\eta_i^*)A^* = B(\sum_{i=1}^{p} \alpha_i\eta_i\eta_i^*)B^*.$$

Next we define $X := \sum_{i=1}^{p} \alpha_i\eta_i\eta_i^*$. We easy check $X = X^*$ and $X \geq 0$. Lets $X^{1/2}$ be its root. Therefore we have

$$AX^{1/2}X^{1/2}A^* = BX^{1/2}X^{1/2}B^*.$$

□

Proof.

We get

$$AX^{1/2} = BX^{1/2}V$$

where V is unitary matrix ($V := X^{1/2}B^*AX^{-1/2}$). Let v be an eigenvector of V and define $u := X^{1/2}$. Then

$$Au = e^{i\theta}Bu$$

. We use theorem with

$$Q := \begin{bmatrix} e^{i\theta}I & 0 \\ 0 & I \end{bmatrix}.$$

□

In this case ($S = 0, F = 4$) we take $m_j = 1$.
Let $a, b, c > 0$ and $d, f \in \mathbb{C}$, $\psi_1, \psi_2 \in \mathbb{R}$. Define $U, V \in \mathbb{C}^{4\times 2}$ by

$$U := \begin{bmatrix} a & 0 \\ b & b \\ c & ic \\ d & f \end{bmatrix}, \quad , \quad V := \begin{bmatrix} 0 & a \\ b & -b \\ c & -ic \\ e^{i\psi_1} f & e^{i\psi_2} d \end{bmatrix},$$

Suppose U, V are unitary. For example:
set $\gamma := 3 + 3^{1/2}, \beta := 3^{1/2} - 1$. Then $a = (2/\gamma)^{1/2}$, $b = 1/(\gamma)^{1/2} = c$, $d = -(\beta/\gamma)^{1/2}$ $f = (1+i)(1/(\gamma\beta))^{1/2}$, $\psi_1 = -\pi/2, \psi_2 = \pi$

Next, we define matrix

$$M := UV^*$$

Obviously $\overline{\sigma}(M) = 1$. Take $\eta \in \mathbb{C}^2$ such that $||\eta|| = 1$. Of course η is the form

$$\eta = \begin{bmatrix} e^{i\Psi_1}\cos\theta \\ e^{i\Psi_2}\sin\theta \end{bmatrix}.$$

Define $\Psi := \Psi_1 - \Psi_2$. We get

$$\nabla_M = \left\{ \begin{bmatrix} a^2(\cos^2\theta - \sin^2\theta) \\ 4b^2 \sin\theta\cos\theta\cos\Psi \\ 4c^2\sin\theta\cos\theta\sin\Psi \end{bmatrix} \in \mathbb{R}^3 : \Psi, \theta \in \mathbb{R} \right\}$$

Easy calculations show that $0 \notin \nabla_M$, hence $\mu(M) < 1$.

On the other hand, setting $\theta = 0$ and $\theta = \pi/2$ we get $[a^2, 0, 0], [-a^2, 0, 0] \in \nabla_M$.

Therefore $0 \in \text{conv}(\nabla_M)$. We obtain

$$\inf_{D \in \mathbb{D}} \overline{\sigma}(D^{1/2} M D^{-1/2}) = \overline{\sigma}(M) = 1.$$

This counter-example show that if $S + F \geq 4$, there exist matrices M with

$$\inf_{D \in \mathbb{D}} \overline{\sigma}(D^{1/2} M D^{-1/2}) = \overline{\sigma}(M) > \mu(M).$$

Other cases are false, but we won't prove it.

In section 3 we have proved the following

Theorem

The following conditions are equivalnet

1. $\overline{\sigma}(D^{1/2}MD^{-1/2}) < \beta$
2. $\lambda_{max}(D^{1/2}M^*D^{-1/2}D^{1/2}MD^{-1/2}) < \beta^2$
3. $D^{1/2}M^*D^{-1/2}D^{1/2}MD^{-1/2} - \beta^2 I < 0$
4. $M^*DM - \beta^2 D < 0$

where $M \in \mathbb{C}^{n\times n}, \beta > 0, D \in \mathbb{D}$

Next we will prove main theorem in this section:

Theorem

Let $\mathbb{D}_R$ be the set of real, symetric, members of $\mathbb{D}$. If M is real, then

$$\inf_{D \in \mathbb{D}} \overline{\sigma}(D^{1/2} M D^{-1/2}) = \inf_{D_R \in \mathbb{D}_R} \overline{\sigma}(D_R^{1/2} M D_R^{-1/2})$$

Proof.

Let $D \in \mathbb{D}$ with $D = D_r + iD_i$ and suppose that $\overline{\sigma}(D^{1/2} M D^{-1/2}) < \beta$. Then

$$M^T (D_r + iD_i) M - \beta^2 (D_r + iD_i) < 0$$

and therefore

$$M^T D_r M - \beta^2 D_r) < 0$$

. Hence

$$\overline{\sigma}(D_r^{1/2} M D_r^{-1/2}) < \beta$$

.

□