Some Analysable Instances of μ-synthesis

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Będlewo, June 2014

based on

- N. J. Young, *Some Analysable Instances of μ-synthesis*, Operator Theory: Advances and Applications **222** (2012) 351–368.
- J. Agler, Z. A. Lykova, N. J. Young, *The complex geometry of a domain related to μ-synthesis*, arXiv:1403.1960.

Outline

1. The μ-synthesis problem
2. The spectral Nevanlinna-Pick problem—bidisc (Agler, Young, 1999)
3. The structured Nevanlinna-Pick problem—tetrablock (Abouhajar, White, Young, 2007)
4. The Agler-Lykova-Young problem—pentablock (Agler, Lykova, Young, 2014)
5. The spectral/structured Carathéodory-Fejér problem (Huang, Marcantognini, Young, 2006)/(Young, 2008)

- The μ-synthesis problem is an interpolation problem for analytic matrix functions, a generalization of the classical problems of Nevanlinna-Pick and Carathéodory-Fejér.
- The symbol μ denotes a type of cost function that is a refinement of the usual operator norm of a matrix and is motivated by the problem of the robust stabilization of a plant that is subject to structured uncertainty.
- The μ-synthesis problem is to construct an analytic matrix function F on the unit disc
  - satisfying a finite number of interpolation conditions and such that
  - $\mu(F(\lambda)) \leq 1$ for $|\lambda| < 1$.

Let $l, k \in \mathbb{N}$. For a vector subspace $E \subset \mathbb{C}^{l\times k}$ and a matrix $A \in \mathbb{C}^{k\times l}$ put

$$E_A := \{X \in E : \det(\mathbb{I}_k - AX) = 0\}.$$

1. $E_0 = \varnothing$, $\{0\}_A = \varnothing$.
2. There are $E \neq \{0\}$ and $A \neq 0$ such that $E_A = \varnothing$.
3. If $A \neq 0$ then $\|X\| \geq \|A\|^{-1}$ for any $X \in E_A$, where $\|\cdot\|$ denotes the operator norm.
4. In particular, $\inf\{\|X\| : X \in E_A\} > 0$, whenever $A \neq 0$.
5. If $\alpha \in \mathbb{C}_*$ then

$$\inf\{\|X\| : X \in E_{\alpha A}\} = |\alpha|^{-1} \inf\{\|X\| : X \in E_A\}.$$

Definition (Doyle, Stein, 1981)

The structured singular value of a matrix $A \in \mathbb{C}^{k\times l}$ relative to the vector subspace $E \subset \mathbb{C}^{l\times k}$ we denote by $\mu_E(A)$ and define by

$$\mu_E(A) := \begin{cases} \frac{1}{\inf\{\|X\|:X\in E_A\}}, & \text{if } E_A \neq \varnothing, \\ 0, & \text{if } E_A = \varnothing \end{cases}.$$

1. $\mu_E : \mathbb{C}^{k\times l} \to \mathbb{R}_+$. μ_E is u.s.c.
2. $\mu_{\{0\}} \equiv 0$, $\mu_E(0) = 0$.
3. There are $E \neq \{0\}$ and $A \neq 0$ such that $\mu_E(A) = 0$.
4. $\mu_E \leq \|\cdot\| = \mu_{\mathbb{C}^{l\times k}}$.
5. If $E' \subset E'' \subset \mathbb{C}^{l\times k}$, then $\mu_{E'} \leq \mu_{E''}$.
6. $\mu_E(\alpha A) = |\alpha|\mu_E(A)$ for any $\alpha \in \mathbb{C}$, $A \in \mathbb{C}^{k\times l}$.

If $l = k$, $n_1, \ldots, n_s, m_1, \ldots, m_t \in \mathbb{N}$ are such that $\sum_{i=1}^{s} n_i + \sum_{j=1}^{t} m_j = k$,

$$E = \left\{\mathrm{Diag}[z_1\mathbb{I}_{n_1}, \ldots, z_s\mathbb{I}_{n_s}, Z_1, \ldots, Z_t] : z_j \in \mathbb{C},\ Z_j \in \mathbb{C}^{m_j\times m_j}\right\},$$

then

1. $\mu_E(A) = \max\{r(XA) : X \in E,\ \|X\| \le 1\}$,
2. μ_E is continuous,
3. $\mu_{\mathrm{span}\{\mathbb{I}_k\}} = r$, where r is the spectral radius.

Going back to the general case of E,

1. μ_E does not satisfy the triangle inequality,
2. if $l = k$, $\mathbb{I}_k \in E$, then $r \le \mu_E$.

Problem (μ-synthesis)

Given $k, l \in \mathbb{N}$, $E \subset \mathbb{C}^{l\times k}$, $A \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{k\times l})$, $B \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{k\times k})$, $C \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{l\times l})$, construct $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ of the form

$$F = A + BQC \quad \text{for some} \quad Q \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{k\times l}), \tag{1}$$

where $\Omega_{\mu_E} := \{X \in \mathbb{C}^{k\times l} : \mu_E(X) \le 1\}$.

Our setting is

- $l = k$,
- $B(\lambda) = (\lambda - \lambda_1)\ldots(\lambda - \lambda_n)\mathbb{I}_k$, $\lambda \in \mathbb{D}$, for some $n \in \mathbb{N}$, $\lambda_j \in \mathbb{D}$, $j = 1, \ldots, n$,
- $C = \mathbb{I}_k$.

We shall consider only two "extremal" cases of B.

If $\lambda_i \neq \lambda_j$ whenever $i \neq j$ then

$$F \text{ satisfies (1)} \quad \text{iff} \quad F(\lambda_j) = A(\lambda_j), \quad j = 1, \ldots, n.$$

Problem (Nevanlinna-Pick type)

Given $k, n \in \mathbb{N}$, $E \subset \mathbb{C}^{k \times k}$, $\lambda_j \in \mathbb{D}$, $\lambda_i \neq \lambda_j$ whenever $i \neq j$, $W_j \in \Omega_{\mu_E}$, $j = 1, \ldots, n$, construct an $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ such that

$$F(\lambda_j) = W_j, \quad j = 1, \ldots, n.$$

1. For $E = \operatorname{span}\{\mathbb{I}_k\}$ we obtain the spectral Nevanlinna-Pick problem.
2. For $k = 2$, $E = \operatorname{span}\left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right\}$ we obtain the structured Nevanlinna-Pick problem.
3. For $k = 2$, $E = \operatorname{span}\left\{\mathbb{I}_2, \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\right\}$ we obtain the Agler-Lykova-Young problem.

If $\lambda_1 = \cdots = \lambda_n = 0$ then

$$F \text{ satisfies (1)} \quad \text{iff} \quad F^{(j)}(0) = A^{(j)}(0), \quad j = 0, 1, \ldots, n-1.$$

Problem (Carathéodory-Fejér type)

Given $k, n \in \mathbb{N}$, $E \subset \mathbb{C}^{k \times k}$, $V_j \in \mathbb{C}^{k \times k}$, $j = 0, 1, \ldots, n$, $V_0 \in \Omega_{\mu_E}$, construct an $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ such that

$$F^{(j)}(0) = V_j, \quad j = 0, 1, \ldots, n.$$

1. For $E = \operatorname{span}\{\mathbb{I}_k\}$ we obtain the spectral Carathéodory-Fejér problem.
2. For $k = 2$, $E = \operatorname{span}\left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right\}$ we obtain the structured Carathéodory-Fejér problem.

Define additionally

- $\Omega^o_{\mu_E} := \{A \in \mathbb{C}^{k\times k} : \mu_E(A) < 1\}$,
- $\overline{\mathbb{B}}_k := \{A \in \mathbb{C}^{k\times k} : \|A\| \leq 1\}$,
- $\mathbb{B}_k := \{A \in \mathbb{C}^{k\times k} : \|A\| < 1\}$,
- $\Sigma_k := \{A \in \mathbb{C}^{k\times k} : r(A) \leq 1\}$,
- $\Sigma^o_k := \{A \in \mathbb{C}^{k\times k} : r(A) < 1\}$.

If $\mathrm{span}\{\mathbb{I}_k\} \subset E \subset \mathbb{C}^{k\times k}$ then

$$\overline{\mathbb{B}}_k \subset \Omega_{\mu_E} \subset \Sigma_k, \quad \mathbb{B}_k \subset \Omega^o_{\mu_E} \subset \Sigma^o_k.$$

$\Omega^o_{\mu_E}$ is typically an unbounded nonconvex and hitherto unstudied domain, and so the construction of $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ is a challenge. A strategy to find F is

1. To find a dimension-reducing polynomial map
$$\pi : \mathbb{C}^{k\times k} \to \mathbb{C}^r$$
with $\pi^{-1}(\pi(\Omega_{\mu_E})) = \Omega_{\mu_E}$ and $r < k^2$.
2. To construct an interpolating function $h \in \mathcal{O}(\mathbb{D}, \pi(\Omega_{\mu_E}))$ for $\pi(\Omega_{\mu_E})$, i.e. function h satisfying
$$h(\lambda_j) = \pi(W_j), \quad j = 1, \ldots, n.$$
The idea is that the geometry of lower-dimensional domain may be more accessible than that of Ω_{μ_E} itself.
3. To lift h modulo π to F, i.e. to construct an analytic lifting F of h.

We shall say that F is an analytic lifting of h if $F \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{2\times 2})$ and $\pi \circ F = h$.
Recall that if F is an analytic lifting of h then $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ since $\pi^{-1}(\pi(\Omega_{\mu_E})) = \Omega_{\mu_E}$.

Recall that $\pi(\overline{\mathbb{B}}_k) \subset \pi(\Omega_{\mu_E})$. If, moreover,

$$\pi(\overline{\mathbb{B}}_k) = \pi(\Omega_{\mu_E})$$

then to get h one may proceed as follows.

- The geometry and the function theory of the Cartan domain $\mathbb{B}_k$ is rich and long established and there are numerous ways of constructing $H \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{B}}_k)$; for example one may use the homogeneity of $\mathbb{B}_k$ to construct an interpolating function H by the standard process of Schur reduction.
- Then $h := \pi \circ H \in \mathcal{O}(\mathbb{D}, \pi(\Omega_{\mu_E}))$. Such an H we shall call a Schur lifting of h.

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The spectral NP problem (bidisc)
The structured NP problem (tetrablock)
The ALY problem (pentablock)
The CF problem
The structured singular value
General form
Our setting
Dimension reduction strategy

1. (Agler, Young, 1999) If $k = 2$, $E = \operatorname{span}\{\mathbb{I}_2\}$ then
$$\mathbb{C}^{2\times 2} \ni A \overset{\pi}{\longmapsto} (\operatorname{tr} A, \det A) \in \mathbb{C}^2,$$
$\pi(\Omega_{\mu_E}) = \pi(\overline{\mathbb{B}}_2) = \overline{\mathbb{G}}_2$, where $\mathbb{G}_2$ is the symmetrized bidisc.
2. (Abouhajar, White, Young, 2007) If $k = 2$, $E = \operatorname{span}\{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\}$ then
$$\mathbb{C}^{2\times 2} \ni A = [a_{ij}] \overset{\pi}{\longmapsto} (a_{11}, a_{22}, \det A) \in \mathbb{C}^3,$$
$\pi(\Omega_{\mu_E}) = \pi(\overline{\mathbb{B}}_2) = \overline{\mathbb{E}}$, where $\mathbb{E}$ is the tetrablock.
3. (Agler, Lykova, Young, 2014) If $k = 2$, $E = \operatorname{span}\{\mathbb{I}_2, \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\}$ then
$$\mathbb{C}^{2\times 2} \ni A = [a_{ij}] \overset{\pi}{\longmapsto} (a_{21}, \operatorname{tr} A, \det A) \in \mathbb{C}^3,$$
$\pi(\Omega_{\mu_E}) = \pi(\overline{\mathbb{B}}_2) = \overline{\mathcal{P}}$, where $\mathcal{P}$ is the pentablock.

We shall briefly discuss the dimension reduction strategy for these instances.

Problem (The spectral Nevanlinna-Pick problem)

Given $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$, $\lambda_i \neq \lambda_j$ whenever $i \neq j$, and $W_1, \ldots, W_n \in \Sigma_k$, construct an $F \in \mathcal{O}(\mathbb{D}, \Sigma_k)$ such that

$$F(\lambda_j) = W_j, \quad j = 1, \ldots, n.$$

For $k = 1$ it reduces to the classical Nevanlinna-Pick problem.

Theorem (Pick, 1916, Nevanlinna, 1919)

Let $\lambda_j, z_j \in \mathbb{D}$, $j = 1, \ldots, n$, $\lambda_i \neq \lambda_j$ whenever $i \neq j$. There is an $F \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ with $F(\lambda_j) = z_j$, $j = 1, \ldots, n$, iff

$$\left[\frac{1 - \bar{z}_i z_j}{1 - \bar{\lambda}_i \lambda_j}\right]_{i,j=1}^n \geq 0,$$

i.e. the left-hand side matrix is nonnegative semidefinite.

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The CF problem
Dimension reduction
Interpolation for $\mathbb{G}$ $(n=2)$
Lifting
Partial solution and the ill-conditioning $(n=2)$
The arbitrary k case

Let $k = 2$ and $E = \operatorname{span}\{\mathbb{I}_2\}$. Recall that the closed and open symmetrized bidiscs are defined by

$$\overline{\mathbb{G}} = \overline{\mathbb{G}}_2 := \{(z + w, zw) : z, w \in \overline{\mathbb{D}}\},$$

$$\mathbb{G} = \mathbb{G}_2 := \{(z + w, zw) : z, w \in \mathbb{D}\}.$$

So here we have

$$\mathbb{C}^{2\times 2} \ni A \overset{\pi}{\longmapsto} (\operatorname{tr} A, \det A) \in \mathbb{C}^2,$$

Recall that $\Omega_{\mu_E} = \Sigma_2$.

- $\mathbb{G}_2 = \pi(\Sigma_2^o)$, $\overline{\mathbb{G}}_2 = \pi(\Sigma_2)$, $\pi^{-1}(\overline{\mathbb{G}}_2) = \Sigma_2$, $\pi^{-1}(\mathbb{G}_2) = \Sigma_2^o$.
- $\mathbb{G}$ is hyperconvex, polynomially convex, starlike about $(0, 0)$, and $\mathbb{C}$-convex, but not convex.
- $l_{\mathbb{G}} = c_{\mathbb{G}}$, while $\mathbb{G}$ cannot be exhausted by domains biholomorphic to convex ones.

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The reason why $\mathbb{G}$ is amenable to analysis is that we have a 1-parameter family of rational functions

$$\Phi_\omega(s,p) = \frac{2\omega p - s}{2 - \omega s}, \quad (s,p) \in \mathbb{G},\ \omega \in \mathbb{T}.$$

We have

Proposition (Agler, Young, 2004)

$\Phi_\omega \in \mathcal{O}(\mathbb{G}, \mathbb{D})$ for any $\omega \in \mathbb{T}$. Conversely, if $(s,p) \in \mathbb{C}^2$ is such that $|\Phi_\omega(s,p)| < 1$ for all $\omega \in \mathbb{T}$, then $(s,p) \in \mathbb{G}$.

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as well as

Proposition (Agler, Young, 2004)

For any $\omega \in \mathbb{T}$, Φ_ω maps $\overline{\mathbb{G}} \setminus \{(2\bar\omega, \bar\omega^2)\}$ analytically into $\overline{\mathbb{D}}$. Conversely, if $(s,p) \in \mathbb{C}^2$ is such that $|\Phi_\omega(rs, r^2p)| < 1$ for all $\omega \in \mathbb{T}$ and $r \in (0,1)$, then $(s,p) \in \overline{\mathbb{G}}$.

Remark

The parameter r is needed: $|\Phi_\omega(s,p)| \leq 1$ for all $\omega \in \mathbb{T}$ is not sufficient - for $p = 1$ the last statement holds true iff $s \in \mathbb{R}$, while for $(s,p) \in \overline{\mathbb{G}}$ there is $|s| \leq 2$.

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We have the following Schwarz Lemma for $\overline{\mathbb{G}}$.

Theorem (Agler, Young, 2001)

Let $\lambda \in \mathbb{D}$, $(s, p) \in \overline{\mathbb{G}}$. The following conditions are equivalent:

1. *There exists an $H \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{G}})$ such that $H(0) = 0$, $H(\lambda) = (s, p)$;*
2. $|s| < 2$ *and* $\frac{2|s - p\bar{s}| + |s^2 - 4p|}{4 - |s|^2} \le |\lambda|$;
3. $\big||\lambda|^2 s - p\bar{s}\big| + |p|^2 + (1 - |\lambda|^2)\frac{|s|^2}{4} - |\lambda|^2 \le 0$;
4. $|s| \le \frac{2}{1 - |\lambda|^2}(|\lambda||1 - p\overline{\omega}^2| - \big||\lambda|^2 - p\overline{\omega}^2\big|)$ *for any* $\omega \in \mathbb{T}$ *with* $s = |s|\omega$.

Observe that if $F \in \mathcal{O}(\mathbb{D}, \Sigma_2)$ solves the spectral NP problem (with $k = 2$), then $H := \pi \circ F$ is an analytic map from $\mathbb{D}$ to $\overline{\mathbb{G}}$, such that

$$H(\lambda_j) = \pi(W_j), \quad j = 1, \dots, n.$$

The problem of conversing the above claim is a little bit more subtle. Namely, we have

Theorem (Agler, Young, 2000)

Let $(\lambda_j, W_j)_{j=1}^n$ be as in spectral NP problem (with $k = 2$). Assume additionally that either all or none of the W_j's are scalar matrices. The following conditions are equivalent:

1. *There exists an $F \in \mathcal{O}(\mathbb{D}, \Sigma_2)$ with $F(\lambda_j) = W_j$, $j = 1, \dots, n$;*
2. *There exists an $H \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{G}})$ with $H(\lambda_j) = \pi(W_j)$, $j = 1, \dots, n$.*

Sketch of proof. It only suffices to deliver the first conditions from the second. Let $H=(H_1,H_2)$ be as in the statement. There are two cases to be considered.
Case 1. The W_j's are nonscalar. Then

$$W_j = P_j^{-1}\begin{bmatrix} 0 & 1 \\ -p_j & s_j \end{bmatrix} P_j, \quad j=1,\dots,n,$$

where $(s_j,p_j)=\pi(W_j)$ and P_j is some nonsingular matrix, $j=1,\dots,n$.
Observe that each P_j has a logarithm L_j. Let L be a matrix polynomial with $L(\lambda_j)=L_j$, $j=1,\dots,n$, and for our purpose it suffices to put

$$F(\lambda) := e^{-L(\lambda)}\begin{bmatrix} 0 & 1 \\ -H_2(\lambda) & H_1(\lambda) \end{bmatrix} e^{L(\lambda)}.$$

Case 2. $W_j=c_j\mathbb{I}_2$, $j=1,\dots,n$. Then $H(\lambda_j)=(2c_j,c_j^2)$ and $|2c_j|\le 2$, $|c_j^2|\le 1$. If now some of the c_j's lies on the unit circle, then by the maximum principle all of the W_j's are equal and we may choose F to be a constant function. In the remaining case observe that $\|H_1\|_\infty\le 2$ and putting $F(\lambda):=\frac{1}{2}H_1(\lambda)\mathbb{I}_2$ finishes the proof.
The assumption concerning the structure of the data matrices is essential. Indeed, we have

Example

Let $\lambda_1=0$, $\lambda_2=\beta\in(0,1)$, $W_1=0$, and $W_2=\begin{bmatrix} 0 & 1 \\ 0 & \frac{2\beta}{1+\beta}\end{bmatrix}$. Using [Agler, Young, 2001] one can check that the function $H(\lambda)=\left(\frac{2\lambda(1-\beta)}{1-\beta\lambda},\frac{\lambda(\lambda-\beta)}{1-\beta\lambda}\right)$ fulfills the second condition of the theorem. However, there is no F as in the first one.

To see this, suppose there is such an F. Then we may write $F(\lambda) = \lambda G(\lambda)$. By theorem of Vesentini, the function $\lambda \mapsto r(G(\lambda))$ is subharmonic. Using now the maximum principle one gets

$$\sup_{|\lambda| \leq t} r(G(\lambda)) = \sup_{|\lambda| = t} \frac{1}{t} r(F(\lambda)) \leq \frac{1}{t},$$

for $t \in (0, 1)$. Therefore $G \in \mathcal{O}(\mathbb{D}, \Sigma_2)$. On the other hand, the eigenvalues of $G(\beta)$ are 0 and $\frac{2}{1+\beta} > 1$, which is nonsense.

The importance of the above example lies in the fact, that is shows that the spectral NP problem can be ill-posed, meaning it can admit no solution. We shall discuss this issue later more detailed.

As we have seen, the interpolation into $\overline{\mathbb{G}}$ is equivalent to the interpolation into Σ_2, unless one of the data matrices is scalar, while the second is not. However, in the latter case the interpolation into Σ_2 is equivalent to interpolation into $\overline{\mathbb{G}}$ with some differential condition.

Theorem (Agler, Young, 2000)

Let $\lambda_1, \lambda_2 \in \mathbb{D}$, $W_1, W_2 \in \Sigma_2$, where $W_1 = c\mathbb{I}_2$ and W_2 is nonscalar. The following statements are equivalent

1. *there exists an $F \in \mathcal{O}(\mathbb{D}, \Sigma_2)$ such that $F(\lambda_j) = W_j$, $j = 1, 2$;*
2. *there exists an $H \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{G}})$ such that $H(\lambda_j) = \pi(W_j)$, $j = 1, 2$, and $H_2'(\lambda_1) = cH_1'(\lambda_1)$.*

Sketch of proof. Let an F as in the first condition. Then all coefficients of the matrix function $F - c\mathbb{I}_2$ vanish at λ_1, which yields $\det(F - c\mathbb{I}_2)$ has a zero of order at least 2 at λ_1. We define

$$H := \pi \circ F$$

and simple calculation reveals it is good for our purpose.
The other implication is a little bit more complicated. Let an H be as in the second statement. We know that

$$W_2 = P^{-1} \begin{bmatrix} 0 & 1 \\ -p & s \end{bmatrix} P,$$

where $(s,p) = \pi(W_2)$ and P is nonsingular. We have three cases to consider.

Case 1. $c = 0$. Then we may write

$$H_2(\lambda) = (\lambda - \lambda_1) g(\lambda),$$

where g is analytic on $\mathbb{D}$ and $g(\lambda_1) = 0$. Define

$$F(\lambda) = P^{-1} \begin{bmatrix} 0 & \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} \\ -(\lambda_2 - \lambda_1) g(\lambda) & H_1(\lambda) \end{bmatrix} P.$$

Using the fact that the characteristic polynomial of $F(\lambda)$ is

$$z^2 - H_1(\lambda) z + H_2(\lambda)$$

and that $H(\lambda) \in \overline{\mathbb{G}}$, one easily verifies that F fulfills the first statement.

Case 2. $c \in \mathbb{D}$. Let

$$h = B_c \circ H : \mathbb{D} \to \overline{\mathbb{G}}, \quad \widetilde{W_2} = \mu_c(W_2),$$

where $B_c(z + w, zw) = (b_c(z) + b_c(w), b_c(z)b_c(w))$, $z, w \in \overline{\mathbb{D}}$, $b_c(\lambda) = \frac{\lambda - c}{1 - \bar{c}\lambda}$, and $\mu_c(A) := (A - c\mathbb{I}_2)(\mathbb{I}_2 - \bar{c}A)^{-1}$. We have

$$\pi(\widetilde{W_2}) = B_c(s, p) = h(\lambda_2),$$

$$h(\lambda_1) = B_c(2c, c^2) = (0, 0).$$

Also, h is analytic and

$$h_2 = \frac{H_2 - cH_1 + c^2}{1 - \bar{c}H_1 + \bar{c}^2 H_2},$$

which yields $h_2'(\lambda_1) = 0$. Making use of Case 1, we find an $\widetilde{h} \in \mathcal{O}(\mathbb{D}, \Sigma_2)$ with $\widetilde{h}(\lambda_1) = 0$ and $\widetilde{f}(\lambda_2) = \widetilde{W_2}$. It is now enough to put $F := \mu_{-c} \circ \widetilde{h}$.

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Case 3. $c \in \mathbb{T}$. Then, since $H(\lambda_1) = (2c, c^2)$, by the maximum principle we conclude that H_1 and H_2 are constant. Therefore, $\operatorname{tr} W_2 = 2c$, $\det W_2 = c^2$. Furthermore,

$$W_2 = R^{-1} \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} R$$

with a nonsingular R. To the end we are looking out for, it suffices to choose an analytic g of $\mathbb{D}$ with $g(\lambda_1) = 0$ and $g(\lambda_2) = 1$, and put

$$F(\lambda) = R^{-1} \begin{bmatrix} c & g(\lambda) \\ 0 & c \end{bmatrix} R.$$

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Theorem (Agler, Young, 2000)

Let $\lambda_1, \lambda_2 \in \mathbb{D}$ and $W_1, W_2 \in \Sigma_2$, where $W_1 = c\mathbb{I}_2$ for a $c \in \mathbb{D}$. Then there exists an $F \in \mathcal{O}(\mathbb{D}, \Sigma)$ such that $F(\lambda_j) = W_j$, $j = 1, 2$, iff

$$r(\mu_c(W_2)) \le \left| \frac{\lambda_1 - \lambda_2}{1 - \overline{\lambda_2}\lambda_1} \right| =: m(\lambda_1, \lambda_2).$$

Observe that the last condition is equivalent to saying that

$$\max \left\{ \left| \frac{\xi_1 - c}{1 - \bar{\xi}_1 c} \right|, \left| \frac{\xi_2 - c}{1 - \bar{\xi}_2 c} \right| \right\} \le m(\lambda_1, \lambda_2),$$

where ξ_1, ξ_2 are the eigenvalues of W_2.

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Theorem (Agler, Young, 2000)

Let $\beta \in \mathbb{D}_*$ and $W_1, W_2 \in \Sigma_2$. Assume that W_1 has a unique eigenvalue, say $c \in \mathbb{D}$. Put $(s, p) = \pi(W_2)$.

1. *If both or neither of the W_j's are scalar matrices, then there exists an $F \in \mathcal{O}(\mathbb{D}, \Sigma_2)$ with $F(0) = W_1, F(\beta) = W_2$ iff* $\frac{2|s - \bar{s}p - 2c(1-|p|^2) + c^2(\bar{s} - s\bar{p})| + (1-|c|^2)|s^2 - 4p|}{|2 - \bar{c}s|^2 - |s - 2\bar{c}p|^2} \le |\beta|$.
2. *If W_1 is scalar, while W_2 is not, then an F as above exists iff*

$$\begin{aligned} 2|\beta||(-2\bar{c}p + (1+|c|^2)s - 2c)(1 - c\bar{s} + c^2\bar{p}) \\ - |\beta|^{-2}(-2\bar{p}c + (1+|c|^2)\bar{s} - 2\bar{c})(p - cs + c^2)| \\ + (1-|c|^2)^2|s^2 - 4p| + 4(1-|\beta|^2)|1 - \bar{c}s + \bar{c}^2 p|^2 \\ \le (1-|c|^2)(|2 - \bar{c}s|^2 - |s - 2\bar{c}p|^2). \end{aligned}$$

Theorem (Agler, Young, 2000)

Let $\lambda_1, \lambda_2 \in \mathbb{D}, W \in \Sigma_2$, and $c \in \mathbb{T}$. Then, there exists an $F \in \mathcal{O}(\mathbb{D}, \Sigma_2)$ such that $F(\lambda_1) = c\mathbb{I}_2$, $F(\lambda_2) = W$ iff c is the only eigenvalue of W.

Let F be a solution of spectral NP problem (with $k = 2$ and arbitrary n). Put $(s_j, p_j) = \pi(W_j)$, $j = 1, \ldots, n$. For any $\omega \in \mathbb{T}$ and $t \in (0, 1)$, the composition

$$\Phi_\omega \circ \pi \circ tF$$

is an analytic self map of $\mathbb{D}$ which sends λ_j to $\Phi_\omega(ts_j, t^2p_j) = \frac{2\omega t^2 p_j - ts_j}{2 - \omega ts_j}$, $j = 1, \ldots, n$. Hence, by Pick's theorem,

$$\left[\frac{1 - \overline{\Phi}_\omega(ts_i, t^2p_i)\Phi_\omega(ts_j, t^2p_j)}{1 - \bar{\lambda}_i\lambda_j}\right]_{i,j=1}^n \geq 0.$$

Conjugating the above matrix by $[(2 - \omega ts_j)\delta_{ji}]_{i,j=1}^n$ (δ_{ji} stands for the Kronecker delta) and putting $\alpha = t\omega$ we have delivered the following necessary condition for the solvability of a 2×2 spectral NP problem.

Theorem (Agler, Young, 1999)

Let F be a solution of spectral NP problem (with $k=2$ and arbitrary n). Put $(s_j,p_j)=\pi(W_j)$, $j=1,\dots,n$. Then for any $\alpha\in\overline{\mathbb{D}}$ we have

$$\left[\frac{\overline{(2-\alpha s_i)}(2-\alpha s_j)-|\alpha|^2\overline{(2\alpha p_i-s_i)}(2\alpha p_j-s_j)}{1-\bar\lambda_i\lambda_j}\right]_{i,j=1}^n\geq 0.$$

In general, the condition given above is not sufficient for the solvability of the 2×2 spectral NP problem as the following example shows.

Let $r\in(0,1)$ and let

$$h(\lambda)=\left(2(1-r)\frac{\lambda^2}{1+r\lambda^3},\frac{\lambda(\lambda^3+r)}{1+r\lambda^3}\right).$$

Let $\lambda_1,\lambda_2,\lambda_3\in\mathbb{D}$ be any three distinct points and let $h(\lambda_j)=(s_j,p_j), j=1,2,3$. Then, using (Agler, Lykova, Young, 2012), one can prove that in any neighbourhood of (s_1,s_2,s_3) in $(2\mathbb{D})^3$ there exists a point (s_1',s_2',s_3') such that $(s_j',p_j)\in\mathbb{G}$, the interpolation data

$$\lambda_j\mapsto(s_j',p_j),\quad j=1,2,3,$$

satisfy the necessary condition of the Theorem, and yet there is no function $H\in\mathcal{O}(\mathbb{D},\overline{\mathbb{G}})$ such that $H(\lambda_j)=(s_j',p_j)$, $j=1,2,3$.
In the case $n=k=2$ however, the condition in the Theorem is also sufficient for the solvability of spectral NP problem.

Theorem (Agler, Young, 2004)

Let $\lambda_1, \lambda_2 \in \mathbb{D}$, let $W_1, W_2 \in \Sigma_2$ be nonscalar, and let $(s_j, p_j) = \pi(W_j)$, $j = 1, 2$. The following statements are equivalent

1. *there exists an $F \in \mathcal{O}(\mathbb{D}, \Sigma_2)$ such that $F(\lambda_j) = W_j$, $j = 1, 2$;*
2. $\max_{\omega\in\mathbb{T}} \left| \frac{(s_2p_1-s_1p_2)\omega^2+2(p_2-p_1)\omega+s_1-s_2}{(s_1-\bar{s}_2p_1)\omega^2-2(1-p_1\bar{p}_2)\omega+\bar{s}_2-s_1\bar{p}_2} \right| \leq m(\lambda_1, \lambda_2)$;
3. *for all $\omega \in \mathbb{T}$ the matrix*

$$\left[\frac{\overline{(2 - \omega s_i)}(2 - \omega s_j) - \overline{(2\omega p_i - s_i)}(2\omega p_j - s_j)}{1 - \bar{\lambda}_i \lambda_j} \right]_{i,j=1}^{n}$$

is nonnegative semidefinite.

Sketch of proof. Recall that for a domain $\Omega \subset \mathbb{C}^k$ the Lempert function $l_\Omega : \Omega^2 \to \mathbb{R}^+$ is defined as

$$l_\Omega(z_1, z_2) := \inf m(\lambda_1, \lambda_2),$$

where infimum is taken over all $\lambda_1, \lambda_2 \in \mathbb{D}$ such that there exists an $h \in \mathcal{O}(\mathbb{D}, \Omega)$ sending λ_j to z_j, $j = 1, 2$.
After (Agler, Young, 2004), we define the Carathéodory distance $C_\Omega : \Omega^2 \to \mathbb{R}^+$ by

$$C_\Omega(z_1, z_2) := \sup m(f(z_1), f(z_2)),$$

where the supremum is taken over all $f \in \mathcal{O}(\Omega, \mathbb{D})$, i.e. we omit the $\tanh^{-1}$ on the right-hand side.

Let $z_j = (s_j, p_j) \in \mathbb{G}$.
(1)⇔(2). We only have to show that the inequality in (2) is equivalent to the existence of an $H \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{G}})$ such that $H(\lambda_j) = z_j$, $j = 1, 2$. By definition of the Lempert function, such an H exists iff $l_{\mathbb{G}}(z_1, z_2) \leq m(\lambda_1, \lambda_2)$. Using (Agler, Young, 2004), we get

$$l_{\mathbb{G}}(z_1, z_2) = C_{\mathbb{G}}(z_1, z_2) = \max_{\omega \in \mathbb{T}} m(\Phi_\omega(z_1), \Phi_\omega(z_2))$$

$$= \max_{\omega \in \mathbb{T}} \left| \frac{(s_2 p_1 - s_1 p_2)\omega^2 + 2(p_2 - p_1)\omega + s_1 - s_2}{(s_1 - \bar{s}_2 p_1)\omega^2 - 2(1 - p_1 \bar{p}_2)\omega + \bar{s}_2 - s_1 \bar{p}_2} \right| \leq m(\lambda_1, \lambda_2).$$

(2)⇔(3). By what we have just proved, the condition (2) is equivalent to

$$\max_{\omega \in \mathbb{T}} m(\Phi_\omega(z_1), \Phi_\omega(z_2)) \leq m(\lambda_1, \lambda_2).$$

Using the Schwarz-Pick lemma we conclude that the above inequality holds true iff for any $\omega \in \mathbb{T}$ there is a function $f_\omega \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with $f_\omega(\lambda_j) = \Phi_\omega(z_j)$, $j = 1, 2$. This latter, by Pick's theorem is equivalent to

$$\left[\frac{1 - \overline{\Phi}_\omega(z_i)\Phi_\omega(z_j)}{1 - \bar{\lambda}_i \lambda_j} \right]_{i,j=1}^2 \geq 0,$$

from which one easily derives the conclusion.

Remark

If we drop the structural assumption about the data matrices, we also have a solvability criterion: if both W_j's are scalar, then the problem reduces to the one-dimensional one. Also, we already know the required criterion in the case $W_1 = c\mathbb{I}_2$ and W_2 nonscalar. Recall that then the corresponding spectral NP problem is solvable iff $r(\mu_c(W_2)) \leq m(\lambda_1, \lambda_2)$.

Observe that the spectral NP problem never has a unique solution. For if F is such a solution, then so is $P^{-1}FP$ for any $P \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{k\times k})$ such that the values of P are nonsingular matrices and $P(\lambda_j)$ is a scalar matrix for each interpolant λ_j.
On the other hand, the solution of the corresponding problem of the interpolation into $\overline{\mathbb{G}}$ can be unique. In fact, such a solution (provided it exists) is unique iff each pair of distinct points of $\mathbb{G}$ lies on a unique complex geodesic of $\mathbb{G}$, and the latter is true by (Agler, Young, 2004) and (Agler, Young, 2006).

We have already mentioned that the spectral NP problem can be ill-posed. In fact, it can admit no solution even if there are arbitrarily close data admitting a solution.

Example

Let $\lambda_1 = 0, \lambda_2 = \beta \in (0,1), \alpha \in \mathbb{C}$,

$$W_1(\alpha) = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 1 \\ 0 & \frac{2\beta}{1+\beta} \end{bmatrix}.$$

We already know that the case $\alpha = 0$ has no solution. On the other hand, for $\alpha \neq 0$, the assumptions of Theorem are fulfilled, and

$$H(\lambda) = \left(\frac{2\lambda(1-\beta)}{1-\beta\lambda}, \frac{\lambda(\lambda-\beta)}{1-\beta\lambda} \right)$$

satisfies its second condition.

Nevertheless, we can give a precise answer to the question: for which $\eta \in \mathbb{D}$ there is an $F \in \mathcal{O}(\mathbb{D}, \Sigma_2)$ such that

$$F(0) = W_1(\alpha), \quad F(\eta) = W_2.$$

It is as follows:

1. If $\alpha \neq 0$, then F exists iff $|\eta| \geq \beta$.
2. If $\alpha = 0$, then F exists iff $|\eta| \geq \frac{2\beta}{1+\beta}$.

Currently, we glimpse at the difficulties which appear when passing to the more general case, with arbitrary k.
There is an obvious way to generalize the symmetrized bidisc. Namely, we define the open symmetrized polydisc $(k \geq 2)$

$$\mathbb{G}_k := \{(\sigma_1(z), \dots, \sigma_k(z)) : z \in \mathbb{D}^k\} \subset \mathbb{C}^k,$$

where σ_j denotes the elementary symmetric polynomial in $z = (z^1, \dots, z^k)$ for $j = 1, \dots, k$. Along the same lines one defines the closed symmetrized polydisc $\overline{\mathbb{G}}_k$.
As in the case $k = 2$, we can reduce the spectral NP problem to an interpolation problem into $\overline{\mathbb{G}}_k$ under some hypotheses on the target matrices W_j. Namely, we need to assume that they are nonderogatory (this means that each eigenvalue of W_j has geometric multiplicity exactly one, i.e. the dimension of corresponding eigenspace is one).

In the case $k = 2$, a matrix A is nonderogatory exactly when it is nonscalar. Also, we have used the fact that a matrix A is nonderogatory iff it is similar to the companion matrix of its characteristic polynomial (Horn, Johnson, 1990).

Two basic problems appear while discussing the relations between the interpolation into Σ_k and into $\overline{\mathbb{G}}_k$.

1. For $k > 2$ there is no such a simple characterization of nonderogatory matrices (cf. (Nikolov, Pflug, Thomas, 2010) for the case $k = 3$).
2. It is not true that $l_{\mathbb{G}_k} = C_{\mathbb{G}_k}$ for $k > 2$ (Nikolov, Pflug, Zwonek, 2007).

In the engineering literature the space E is usually taken to be given by a block diagonal structure. If we confine ourselves to $k = 2$ it is natural to study the space of diagonal matrices

$$E = \operatorname{span}\{\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right]\}.$$

Put

$$\mathbb{C}^{2\times 2} \ni A = [a_{ij}] \stackrel{\pi}{\longmapsto} (a_{11}, a_{22}, \det A) \in \mathbb{C}^3$$

and let $\mathbb{E} := \pi(\mathbb{B}_2)$ denote the tetrablock.

- $\mathbb{E}$ is a polynomially convex, $\mathbb{C}$-convex and pseudoconvex bounded domain in $\mathbb{C}^3$.
- $\mathbb{E}$ is starlike about 0, non-circular but $(1,0,1)$-, $(0,1,1)$- and $(1,1,2)$-balanced.
- $\mathbb{E} \cap \mathbb{R}^3$ is a regular tetrahedron with vertices $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$.
- $l_{\mathbb{E}} = c_{\mathbb{E}}$ although $\mathbb{E}$ cannot be exhausted by domains biholomorphic to convex ones.

- $\mathbb{E} = \pi(\Omega^o_{\mu_E})$, $\overline{\mathbb{E}} = \pi(\Omega_{\mu_E})$, $\pi^{-1}(\overline{\mathbb{E}}) = \Omega_{\mu_E}$.

But the true reason that $\mathbb{E}$ is amenable to analysis is that, as in the case of $\mathbb{G}_2$, there is 1-parameter family of rational functions

$$\Psi_\eta(z_1,z_2,z_3) := \begin{cases} \frac{\eta z_3 - z_2}{\eta z_1 - 1}, & \text{if } \eta z_1 \neq 1 \\ z_2, & \text{if } z_1 z_2 = z_3 \end{cases},$$

where $\eta \in \mathbb{C}$, $(z_1,z_2,z_3) \in \mathbb{C}^3$ is such that $\eta z_1 \neq 1$ or $z_1 z_2 = z_3$.

Theorem (Abouhajar, White, Young, 2007)

Let $z \in \mathbb{C}^3$. Then the following are equivalent

1. $z \in \mathbb{E}$;
2. $|\Psi_\eta(z)| < 1$ *for any* $\eta \in \overline{\mathbb{D}}$.

In particular, $\Psi_\eta \in \mathcal{O}(\mathbb{E},\mathbb{D})$ *for any* $\eta \in \overline{\mathbb{D}}$.

Solution of the interpolation problem for $\mathbb{E}$ and $n=2$ is the following Schwarz Lemma for $\mathbb{E}$

Theorem (Abouhajar, White, Young, 2007)

Let $\lambda \in \mathbb{D}_*$, $z=(a,b,p) \in \mathbb{E}$. Then the following are equivalent

1. *there is* $h \in \mathcal{O}(\mathbb{D},\mathbb{E})$, *with* $h(0)=0$, $h(\lambda)=z$;
2. *there is* $h \in \mathcal{O}(\mathbb{D},\overline{\mathbb{E}})$, *with* $h(0)=0$, $h(\lambda)=z$;
3. $\max\left\{\frac{|a-\bar{b}p|+|ab-p|}{1-|b|^2}, \frac{|b-\bar{a}p|+|ab-p|}{1-|a|^2}\right\} \leq |\lambda|$;
4. *either*
   - $|b| \leq |a|$ *and* $\frac{|a-\bar{b}p|+|ab-p|}{1-|b|^2} \leq |\lambda|$ *or*
   - $|a| \leq |b|$ *and* $\frac{|b-\bar{a}p|+|ab-p|}{1-|a|^2} \leq |\lambda|$;
5. *there is* $H \in \mathcal{O}(\mathbb{D},\overline{\mathbb{B}}_2)$ *with* $H(0) \in \pi^{-1}(0)$, $H(\lambda) \in \pi^{-1}(z)$.

The interpolation problems for Ω_{μ_E} and $\overline{\mathbb{E}}$ are equivalent in the following sense.

Theorem (Abouhajar, White, Young, 2007)

Let $\lambda_j \in \mathbb{D}$, $\lambda_i \neq \lambda_j$ whenever $i \neq j$, $W_j \in \Omega^o_{\mu_E}$ (resp. $W_j \in \Omega_{\mu_E}$), $j = 1, \ldots, n$. Then the following are equivalent

1. *there is $F \in \mathcal{O}(\mathbb{D}, \Omega^o_{\mu_E})$ (resp. $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$) with $F(\lambda_j) = W_j$, $j = 1, \ldots, n$;*
2. *there is $h \in \mathcal{O}(\mathbb{D}, \mathbb{E})$ (resp. $h \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{E}})$) with $h(\lambda_j) = \pi(W_j)$, $j = 1, \ldots, n$, and, if $W_j = [w^j_{st}]$ is diagonal, then*
$$h_3'(\lambda_j) = w^j_{22} h_1'(\lambda_j) + w^j_{11} h_2'(\lambda_j).$$

On putting together three previous theorems one get the partial solution of the structured Nevanlinna-Pick problem for $n = 2$.

Theorem (Abouhajar, White, Young, 2007)

Let $\lambda \in \mathbb{D}_*$, $A_1, A_2 \in \Omega_{\mu_E}$, where $\pi(A_1) = 0$, $\pi(A_2) = (a, b, p)$, and $A_2 \notin E$. Then the following are equivalent

1. *there is $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ with $F(0) = A_1$, $F(\lambda) = A_2$;*
2. $$\begin{cases} \max\left\{\frac{|a-\bar{b}p|+|ab-p|}{1-|b|^2}, \frac{|b-\bar{a}p|+|ab-p|}{1-|a|^2}\right\} \leq |\lambda|, & \text{if } A_1 \neq 0 \\ \left(\frac{a}{\lambda}, \frac{b}{\lambda}, \frac{p}{\lambda^2}\right) \in \overline{\mathbb{E}}, & \text{if } A_1 = 0 \end{cases}.$$

- The solvability of structured Nevanlinna-Pick problem is equivalent to the calculation of $l_{\mathbb{E}}$.
- If $z = (a, b, p) \in \mathbb{E}$ then
$$l_{\mathbb{E}}(0, z) = \max\left\{\tanh^{-1} \frac{|a-\bar{b}p|+|ab-p|}{1-|b|^2}, \tanh^{-1} \frac{|b-\bar{a}p|+|ab-p|}{1-|a|^2}\right\}.$$
- What about $l_{\mathbb{E}}(w, z)$ for $w \neq 0$? It suffices to consider $w = (0, 0, \alpha)$, $0 < \alpha < 1$.

Using the form of automorphisms of $\mathbb{E}$ and the previous theorems we may get the following

Theorem (Abouhajar, White, Young, 2007)

Let $\lambda_1, \lambda_2 \in \mathbb{D}$, $\lambda_1 \neq \lambda_2$, $A_1, A_2 \in \Omega_{\mu_E}$, where $\pi(A_1) = z$, $\pi(A_2) = w$, and A_1 is triangular. Then the following are equivalent

1. *there is $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ with $F(\lambda_1) = A_1$, $F(\lambda_2) = A_2$;*
2. $\max\{\alpha(z,w), \alpha(\tilde{z},\tilde{w})\} \leq \left|\frac{\lambda_1-\lambda_2}{1-\lambda_1\bar{\lambda}_2}\right|$, *where*
   - $\alpha(z,w) := \frac{(1-|z_1|^2)\beta(w)+|\gamma(w)-\delta(w)z_1+\epsilon(w)z_1^2|}{|1-\bar{z}_1w_1|^2-|w_2-\bar{z}_1w_3|^2}$,
   - $\beta(w) = |w_3 - w_1w_2|$,
   - $\gamma(w) = w_1 - \bar{w}_2w_3$,
   - $\delta(w) = 1 + |w_1|^2 - |w_2|^2 - |w_3|^2$,
   - $\epsilon(w) = \bar{w}_1 - w_2\bar{w}_3$,
   - $\tilde{x} = (x_2, x_1, x_3)$ *for any* $x = (x_1, x_2, x_3) \in \mathbb{C}^3$.

Take in the second last theorem above $(a,b,p) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Since $\pi(A_1) = 0$ there is $\zeta \in \mathbb{C}$ such that

$$A_1 = \begin{bmatrix} 0 & \zeta \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A_1 = \begin{bmatrix} 0 & 0 \\ \zeta & 0 \end{bmatrix}.$$

Then there is $F_\zeta \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ with $F_\zeta(0) = A_1$, $F_\zeta(\lambda) = A_2$ iff

$$|\lambda| \geq \begin{cases} \frac{2}{3}, & \text{if } A_1 \neq 0 \\ \frac{1}{\sqrt{2}}, & \text{if } A_1 = 0 \end{cases}.$$

It follows that if $\frac{2}{3} < |\lambda| < \frac{1}{\sqrt{2}}$ then F_ζ cannot be locally bounded as $\zeta \to 0$. For such λ, if ζ is close to zero then the solutions of the interpolation problem are very sensitive to small changes in ζ.

Recently Agler, Lykova, and Young started the investigation of the μ-synthesis problem related to the space

$$E = \operatorname{span}\left\{\mathbb{I}_2, \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]\right\},$$

another natural choice of E. Put

$$\mathbb{C}^{2\times 2} \ni A = [a_{ij}] \overset{\pi}{\longmapsto} (a_{21}, \operatorname{tr} A, \det A) \in \mathbb{C}^3,$$

and let $\mathcal{P} := \pi(\mathbb{B}_2)$ denote the pentablock.

- $\mathcal{P}$ is a polynomially convex, non-convex bounded domain.
- $\mathcal{P}$ is starlike about 0, non-circular but $(1,0,0)$- and $(k,1,2)$-balanced, $k \geq 0$.
- $\mathcal{P} \cap \mathbb{R}^3$ in a convex body bounded by five faces, comprising two triangles, an ellipse and two curved surfaces, with four vertices $(0,-2,1)$, $(0,2,1)$, $(1,0,-1)$, and $(-1,0,-1)$.
- $\mathcal{P} = \pi(\Omega^o_{\mu_E})$, $\overline{\mathcal{P}} = \pi(\Omega_{\mu_E})$, $\pi^{-1}(\overline{\mathcal{P}}) = \Omega_{\mu_E}$.

The μ-synthesis problem
The spectral NP problem (bidisc)
The structured NP problem (tetrablock)
The ALY problem (pentablock)
The CF problem
Dimension reduction
Interpolation for $\overline{\mathcal{P}}$ $(n=2)$
Lifting

Again, there is 1-parameter family of rational functions $(\eta \in \mathbb{D})$

$$\Psi_\eta(z) := \frac{(1-|\eta|^2)z_1}{1-\eta z_2 + \eta^2 z_3}, \quad z = (z_1, z_2, z_3) \in \mathbb{C}^3,\ 1 \neq \eta z_2 - \eta^2 z_3.$$

Theorem (Agler, Lykova, Young, 2014)

Let $a \in \mathbb{C}$, $s = \lambda_1 + \lambda_2$, $p = \lambda_1\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{D}$. Put $z = (a, s, p)$. Then the following are equivalent

1. $z \in \mathcal{P}$;
2. $|a| < \left|1 - \frac{s\bar{\beta}/2}{1+\sqrt{1-|\beta|^2}}\right|$, *where* $\beta := \frac{s - \bar{s}p}{1-|p|^2}$;
3. $|a| < \frac{1}{2}\left(|1-\lambda_1\bar{\lambda}_2| + \sqrt{(1-|\lambda_1|^2)(1-|\lambda_1|^2)}\right)$;
4. $\sup_{\eta\in\mathbb{D}} |\Psi_\eta(z)| < 1$.

In particular, $\Psi_\eta \in \mathcal{O}(\mathcal{P}, \mathbb{D})$ for any $\eta \in \mathbb{D}$.

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Some Analysable Instances of μ-synthesis

What is the Schwarz Lemma for $\mathcal{P}$, i.e for which pairs $\lambda \in \mathbb{D}_*$ and $z \in \mathcal{P}$ does there exist $h \in \mathcal{O}(\mathbb{D}, \mathcal{P})$ such that $h(0) = 0$ and $h(\lambda) = z$? A necessary condition is the following

Theorem (Agler, Lykova, Young, 2014)

Let $\lambda \in \mathbb{D}_*$ and $z = (a, s, p) \in \mathcal{P}$. If $h \in \mathcal{O}(\mathbb{D}, \mathcal{P})$ satisfies $h(0) = 0$ and $h(\lambda) = z$, then

$$\max \left\{ \frac{2|s - s\bar{s}p| + |s^2 - 4p|}{4 - |s|^2}, \frac{|a|}{\left|1 - \frac{s\bar{\beta}/2}{1 + \sqrt{1 - |\beta|^2}}\right|} \right\} \leq |\lambda|,$$

where

$$\beta = \frac{s - \bar{s}p}{1 - |p|^2}.$$

On dividing through λ the above inequality and letting $\lambda \to 0$ we obtain an infinitesimal necessary condition.

Corollary (Agler, Lykova, Young, 2014)

If $h = (h_1, h_2, h_3) \in \mathcal{O}(\mathbb{D}, \mathcal{P})$ satisfies $h(0) = 0$, then

$$|h_1'(0)| \leq 1, \quad \frac{1}{2}|h_2'(0)| + |h_3'(0)| \leq 1.$$

Is there a converse? Is it the case that if

$$|z_1| \leq 1, \quad \frac{1}{2}|z_2| + |z_3| \leq 1 \tag{2}$$

then there is $h = (h_1, h_2, h_3) \in \mathcal{O}(\mathbb{D}, \overline{\mathcal{P}})$ such that $h(0) = 0$ and $h'(0) = (z_1, z_2, z_3)$?
The answer is no.

Take $z_1 = 1$, $0 < z_3 < 1$, $z_2 = 2(1 - z_3)$. The inequalities (2) hold. Suppose there is $h = (h_1, h_2, h_3) \in \mathcal{O}(\mathbb{D}, \overline{\mathcal{P}})$ such that $h(0) = 0$ and $h'(0) = (z_1, z_2, z_3)$. Since $h_1 \in \mathcal{O}(\mathbb{D}, \mathbb{D})$, $h_1(0) = 0$, and $h_1'(0) = 1$ we infer that $h_1 = \mathrm{id}_{\mathbb{D}}$. Since $\frac{1}{2}|z_2| + |z_3| = 1$, the description of complex geodesics of $\mathbb{G}_2$ tells us that

$$(h_2, h_3)(\lambda) = \frac{\lambda}{1 + z_3\lambda}(2(1 - z_3), \lambda + z_3), \quad \lambda \in \mathbb{D},$$

is unique function $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_2)$ with $\varphi(0) = 0$ and $\varphi'(0) = (z_2, z_3)$. However, $h(\mathbb{D}) \not\subset \overline{\mathcal{P}}$. Indeed, $h(1) = (1, 2\xi, 1)$, where $\xi = \frac{1-z_3}{1+z_3} \in (0, 1)$. For the point $(s, p) = (2\xi, 1)$ we have $\beta = \xi$, and so

$$\left|1 - \frac{s\bar{\beta}/2}{1 + \sqrt{1 - |\beta|^2}}\right| = 1 - \frac{\xi^2}{1 + \sqrt{1 - \xi^2}} = \sqrt{1 - \xi^2} < 1.$$

Hence $h(1) = (1, 2\xi, 1) \notin \overline{\mathcal{P}}$.

The lifting problem for $\mathcal{O}(\mathbb{D}, \mathcal{P})$ is delicate, as the following examples show.

Example (Agler, Lykova, Young, 2014)

Let $h(\lambda) = (\lambda, 0, \lambda)$, $\lambda \in \mathbb{D}$. This $h \in \mathcal{O}(\mathbb{D}, \mathcal{P})$ lifts to Schur lifting $H \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{B}}_2)$ given by

$$H(\lambda) = \begin{bmatrix} 0 & -1 \\ \lambda & 0 \end{bmatrix}.$$

Here $H(\lambda) \notin \mathbb{B}_2$ for any $\lambda \in \mathbb{D}$, since $\|H(\lambda)\| = 1$.
On the other hand, there is non-analytic lifting $H : \mathbb{D} \to \mathbb{B}_2$ of h given by

$$H(\lambda) = \begin{bmatrix} i\sqrt{1 - |\lambda|}\zeta & -\lambda \\ \lambda & -i\sqrt{1 - |\lambda|}\zeta \end{bmatrix},$$

where ζ is a square root of λ.

Example (Agler, Lykova, Young, 2014)

Let $h(\lambda) = (\lambda^2, 0, \lambda)$, $\lambda \in \mathbb{D}$. This $h \in \mathcal{O}(\mathbb{D}, \mathcal{P})$ has no analytic lifting.
Indeed, suppose $H \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{2\times 2})$ is an analytic lifting of h. We can write

$$H(\lambda) = \begin{bmatrix} -\eta(\lambda) & g(\lambda) \\ \lambda^2 & \eta(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{D},$$

for some $g, \eta \in \mathcal{O}(\mathbb{D}, \mathbb{C})$. Since $\det H(\lambda) = \lambda$, we must have

$$(\eta(\lambda))^2 = -\lambda - \lambda^2 g(\lambda), \quad \lambda \in \mathbb{D}.$$

This is a contradiction, since the right hand side has a simple zero at 0, while the left hand side has a zero of multiplicity at least 2.

These examples point to the following result.

Proposition (Agler, Lykova, Young, 2014)

A function $h = (a, s, p) \in \mathcal{O}(\mathbb{D}, \mathcal{P})$ has analytic lifting iff there is no $\alpha \in \mathbb{D}$ such that, for some odd positive integer n,

- $h(\alpha) \in \mathcal{R} := \{(0, 2\lambda, \lambda^2) : \lambda \in \mathbb{C}\}$,
- *α is a zero of $s^2 - 4p$ of multiplicity n, and*
- *α is a zero of a of multiplicity greater than n.*

Example (Agler, Lykova, Young, 2014)

Let $h(\lambda) = (\frac{1}{2}, 0, \lambda)$, $\lambda \in \mathbb{D}$. This $h \in \mathcal{O}(\mathbb{D}, \overline{\mathcal{P}})$ has an analytic lifting but no Schur lifting.

The upshot of the three examples and proposition is that the μ-synthesis problem for μ_E and the interpolation problem for $\mathcal{O}(\mathbb{D}, \overline{\mathcal{P}})$ are quite closely related, but that the rich function theory of $\mathcal{O}(\mathbb{D}, \overline{\mathbb{B}}_2)$ may not be helpful for their solution.

Problem (The spectral Carathéodory-Fejér problem)

Given $V_0, \ldots, V_n \in \mathbb{C}^{k\times k}$, $V_0 \in \Sigma_k$, construct an $F \in \mathcal{O}(\mathbb{D}, \Sigma_k)$ such that

$$F^{(j)}(0) = V_j, \quad j = 0, \ldots, n.$$

For $k = 1$ it reduces to the classical Carathéodory-Fejér problem.

Theorem (Carathéodory, Fejér, 1911)

Let $z_j \in \mathbb{C}$, $j = 0, 1, \ldots, n$, $z_0 \in \overline{\mathbb{D}}$. There is an $F \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ with $F^{(j)}(0) = z_j$, $j = 0, 1, \ldots, n$ iff Toeplitz matrix $T = [t_{ij}]_{i,j=0}^n$, where

$$t_{ij} := \begin{cases} 0, & \text{if } i - j < 0 \\ c_{i-j}, & \text{if } i - j \geq 0 \end{cases},$$

is a contraction, i.e. $\|T\| \leq 1$.

For $k = 2$, $n = 1$ we have the following

Theorem (Huang, Marcantognini, Young, 2006)

Let $V_j = [v_{ik}^j]_{i,k=1}^2 \in \mathbb{C}^{2\times 2}$, $j = 0, 1$, where $V_0 \in \Sigma_2^o$ is nonscalar. The following are equivalent

1. *there is an $F \in \mathcal{O}(\mathbb{D}, \Sigma_2^o)$ with $F(0) = V_0$, $F'(0) = V_1$;*
2. $\max_{\omega\in\mathbb{T}} \left| \frac{(s_1 p_0 - s_0 p_1)\omega^2 + 2\omega p_1 - s_1}{\omega^2(s_0 - \bar{s}_0 p_0) - 2\omega(1-|p_0|^2) + \bar{s}_0 - s_0\bar{p}_0} \right| \leq 1$, *where* $(s_0, p_0) = \pi(V_0)$, $s_1 = \operatorname{tr} V_1$, *and*

$$p_1 = \begin{vmatrix} v_{11}^0 & v_{12}^1 \\ v_{21}^0 & v_{22}^1 \end{vmatrix} + \begin{vmatrix} v_{11}^1 & v_{12}^0 \\ v_{21}^1 & v_{22}^0 \end{vmatrix}.$$

Problem (The structured Carathéodory-Fejér problem)

Given $V_0, \dots, V_n \in \mathbb{C}^{2\times 2}$, $V_0 \in \Omega_{\mu_E}$, construct an $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ such that

$$F^{(j)}(0) = V_j, \quad j = 0, \dots, n.$$

Again the problem can be reduced to an interpolation problem for $\mathbb{E}$, but the resulting problem has only been solved in an exceedingly special case.

Theorem (Young, 2008)

Let $V_0 = \left[\begin{smallmatrix} 0 & \zeta \\ 0 & 0 \end{smallmatrix}\right]$, $\zeta \in \mathbb{C}$, *and let* $V_1 = [v_{ij}]_{i,j=1}^2 \in \mathbb{C}^{2\times 2}$ *be nondiagonal. The following are equivalent*

1. *there is an* $F \in \mathcal{O}(\mathbb{D}, \Omega_{\mu_E})$ *with* $F(0) = V_0$, $F'(0) = V_1$*;*
2. $\max\{|v_{11}|, |v_{22}|\} + |\zeta v_{21}| \le 1.$

The μ-synthesis problem
The spectral NP problem (bidisc)
The structured NP problem (tetrablock)
The ALY problem (pentablock)
The CF problem
The spectral CF problem (bidisc)
The structured CF problem (tetrablock)

Thank You!

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