The Complex Structured Singular Value
Part 8 - Relating μ and $\inf_{D\in\mathbf{D}} \bar{\sigma}(D^{\frac{1}{2}}MD^{-\frac{1}{2}})$
based on paper by A.Pacard and J.Doyle

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We want to charakterize the case when

$$\bar{\sigma}(M) = \inf_{D\in\mathbf{D}} \bar{\sigma}(D^{\frac{1}{2}}MD^{-\frac{1}{2}})$$

Let $M \in \mathbb{C}^{n\times n}$ and let take its singular value decomposition

$$M = \sigma_1 UV^* + U_2\Sigma_2 V_2^*,$$

where σ_1 is the maximum singular value of M with multiplicity r, $U, V \in \mathbb{C}^{n\times r}$, $U_2, V_2 \in \mathbb{C}^{n\times(n-r)}$, $U^*U = V^*V = I_r$, $U_2^*U_2 = V_2^*V_= I_{n-r}$, $U^*U_2 = 0$, $V^*V_2 = 0$ and $\Sigma_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ is nonnegative and diagonal.

Let define

$$\mathbf{Z} := \{D - \tilde{D} : D, \tilde{D} \in \mathbf{D}\}.$$

Elements of $\mathbf{Z}$ are of the form

$$diag[Z_1, \dots, Z_S, z_{S+1}I_{m_1}, \dots, z_{S+F-1}I_{M_{F-1}}, 0_{m_F}],$$

where for $i \leqslant S$, $Z_i = Z_i^* \in \mathbb{C}^{r_i\times r_i}$, and for $j \leqslant F-1$, $z_{S+j} \in \mathbb{R}$.

Let partition U and V compatibly with Δ as

$$U = \begin{bmatrix} A_1 \\ \vdots \\ A_S \\ E_1 \\ \vdots \\ E_F \end{bmatrix}, V = \begin{bmatrix} B_1 \\ \vdots \\ B_S \\ H_1 \\ \vdots \\ H_F \end{bmatrix},$$

where $A_i, B_i \in \mathbb{C}^{r_i\times r}$, $E_i, H_i \in \mathbb{C}^{m_i\times r}$.

Let define $\nabla_M \subset \mathbf{Z}$ by

$$\nabla_M := \{diag[P_1^\eta, \dots, P_S^\eta, p_{S+1}^\eta I_{m_1}, \dots, p_{S+F-1}^\eta I_{m_{F-1}}, 0_{m_F}] : \eta \in \mathbb{C}^r, ||\eta|| = 1\},$$

where

$$P_i^\eta := A_i\eta\eta^*A_i^* - B_i\eta\eta^*B_i^*,$$
$$p_{S+j}^\eta := \eta^*(E_j^*E_j - H_j^*H_j)\eta.$$

Theorem 1. $\bar{\sigma}(M) = \inf_{D\in\mathbf{D}} \bar{\sigma}(D^{\frac{1}{2}}MD^{-\frac{1}{2}})$ if and only if $0 \in co(\nabla_M)$.

Theorem 2. The following conditions are equivalent

(1) $0 \in \nabla_M$;
(2) there exists $\eta \in \mathbb{C}^r, ||\eta|| = 1$ and $Q \in \mathbf{Q}$ such that $QU\eta = V\eta$;
(3) there exists $\xi \in \mathbb{C}^n, ||\xi|| = 1$ and $Q \in \mathbf{Q}$ such that $QU\xi = \sigma_1\xi$;
(4) $\sigma_1 = \bar{\sigma}(M) = \mu_\delta(M)$.

Dowód. (1) $\Rightarrow$ (2) If $0 \in \nabla_M$, for some $\eta \in \mathbb{C}^r, ||\eta|| = 1$ we have

$$(*) A_i \eta \eta^* A_i^* - B_i \eta \eta^* B_i^* = 0, i \leqslant S$$

$$(**) \eta^* (E_j^* E_j - H_j^* H_j) \eta = 0, j \leqslant F - 1.$$

Let denote $A_i \eta = X = \begin{bmatrix} x_1 \\ \vdots \\ x_{r_i} \end{bmatrix}$ and $B_i \eta = Y = \begin{bmatrix} y_1 \\ \vdots \\ y_{r_i} \end{bmatrix}$. Using this notation, $(*)$ has the following form $XX^* = YY^*, i \leqslant S$. Thus $|x_i|^2 = |y_i|^2$ for $i \leqslant S$ and $x_l \bar{x}_k = y_l \bar{y}_k$ for $l, k \leqslant S$. We may assume that $x_1 \neq 0$. Then $x_k = \frac{|\bar{y}_1|}{|\bar{x}_1|} y_k$, so for $k \leqslant S$ there is a phase $e^{k\theta_j}$ such that $e^{k\theta_j} A_k \eta = B_k \eta$.

$(**)$ means that for $j \leqslant F - 1$,

$$< \eta, E_j^* E_j \eta >=< \eta, H_j^* H_j \eta >,$$

thus

$$< E_j \eta, E_j \eta >=< H_j \eta, H_j \eta >$$

and that is equivalent to $||E_j|| = ||H_j||$. So there exists a unitary matrix Q_j such that $Q_j E_j \eta = H_j \eta$. Finally, since $||U\eta|| = ||V\eta||$ we must have $||E_F \eta|| = ||H_F \eta||$. This gives a unitary matrix Q_F such that $Q_F E_F \eta = H_F \eta$. Arranging the phases and Q_S in a block diagonal matrix gives (2).

(2) $\Rightarrow$ (1) Very similar to the proof (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) Since $M = \sigma_1 UV^* + U_2 \Sigma_2 V_2^*$, we have $QMV\eta = \sigma_1 QU\eta = \sigma_1 V\eta$. $\xi = V\eta$ satisfies (3).

(3) $\Rightarrow$ (2) Since $M = \sigma_1 UV^* + U_2 \Sigma_2 V_2^*$, $QM = \sigma_1 QUV^* + QU_2 \Sigma_2 V_2^*$. Let denote $U_2 = [U \quad U_1]$, $V_2 = [V \quad V_1]$, $\Sigma = diag[\sigma_1 I_r, \Sigma_2]$ and $\tilde{\eta} := V_2^* \xi$. Then $U_2^* M = \Sigma V_2^*$. From (3), $||M\xi|| = \sigma_1$. Thus $\sigma_1^2 = ||U_2^* M \xi||^2 = ||\Sigma V_2^* \xi||^2 = ||\Sigma \eta||^2 = \sigma_1^2 |\eta_1|^2 + \ldots + \sigma_n^2 |\eta_n|^2 \leqslant \sigma_1^2 ||\eta||^2 = \sigma_1^2$. This means that for every j, $\sigma_j = \sigma_1$ or $\eta_j = 0$. Thus $\tilde{\eta} = [\eta \quad 0]$ for some $\eta \in \mathbb{C}^r$. This implies $V\eta = \xi$. Then we have

$$QU\eta = QUV^* \xi = \frac{1}{\sigma_1} QM\xi = \xi = V\eta.$$

(3) $\Rightarrow$ (4) $QM\xi = \sigma_1 \xi$ implies that

$$\sigma_1 \geqslant \mu_\Delta(M) = \max_{Q \in \mathbf{Q}} \rho(QM) \geqslant \rho(QM) \geqslant \sigma_1.$$

(2) $\Rightarrow$ (3) Since

$$\max_{Q \in \mathbf{Q}} \rho(QM) = mu_\Delta(M) = \sigma_1,$$

σ_1 is an eigenvalue of QM, which gives (3).

□

Definition 1. We say that a block structure is μ-simple if for all $W \in \mathbb{C}^{n \times n}$, $0 \in co(\nabla_W)$ implies $0 \in \nabla_W$.

Theorem 3. Let block structure Δ be μ-simple. Then, for every $M \in \mathbb{C}^{n \times n}$,

$$\mu_\Delta(M) = \inf_{D \in \mathbf{D}} \bar{\sigma}(D^{\frac{1}{2}} M D^{-\frac{1}{2}}).$$

Dowód. Let $\beta = \inf_{D\in\mathbf{D}} \bar{\sigma}(D^{\frac{1}{2}}MD^{-\frac{1}{2}})$. Let D_k be a sequence in $\mathbf{D}$ such that

$$\bar{\sigma}(D^{\frac{1}{2}}MD^{-\frac{1}{2}}) \longrightarrow \beta, \text{ as } k \longrightarrow \infty.$$

Denote $W_k = D^{\frac{1}{2}}MD^{-\frac{1}{2}}$. Since the sequence W_k is bounded, it has a convergent subsequence with limit W. By continuity of $\bar{\sigma}$ and μ_Δ, $\bar{\sigma}(w) = \beta$ and $\mu_{Delta}(M) = \mu_{Delta}(W)$. We claim that $0 \in co(\nabla_W)$. Suppose to the contrary that $0 \notin co(\nabla_W)$. Then, by Theorem 1., there exist $D \in \mathbf{D}$ end $\epsilon > 0$ such that $\bar{\sigma}(D^{\frac{1}{2}}WD^{-\frac{1}{2}}) = \beta - \epsilon$. Choose k so that

$$||W_k - W|| < \frac{\epsilon}{2||D^{\frac{1}{2}}||||D^{-\frac{1}{2}}||}.$$

Then

$$||D^{\frac{1}{2}}(W_k - W)D^{-\frac{1}{2}}|| < \frac{\epsilon}{2},$$

which yields

$$||D^{\frac{1}{2}}W_kD^{-\frac{1}{2}}|| < \beta - \frac{\epsilon}{2}.$$

This contradicts that β was the infimum, thus indeed $0 \in co(\nabla_W)$. But Δ is μ-simple, so $0 \in \nabla_W$ and by Theorem 2. $\bar{\sigma}(W) = \mu_\delta(W)$. Finally we get $\mu_\delta(M) = \beta$ as desired.

□