Transfer functions, state space tests for robust performance

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June 18, 2014

Let $M \in \mathbb{C}^{(n+n)\times(n+m)}$ be a block matrix.
We define the *transfer function matrix*

$$G(z) = \mathscr{S}(\frac{1}{z} I_n, M) = M_{22} + M_{21}(zI - M_{11})^{-1} M_{12}.$$

Suppose $\Delta \subset \mathbb{C}^{n\times n}$ is some block structure. Put

$$\Delta_P = \left\{ \mathrm{diag}[\delta_1 I_n, \triangle] : \delta_1 \in \mathbb{C}, \triangle \in \Delta \right\}.$$

The following statements are equivalent:

1. $\rho(M_{11}) < 1$ and $\max_{\theta\in[0,2\pi]} \mu_\Delta(G(e^{i\theta})) < 1$;
2. $\rho(M_{11}) < 1$ and $\max_{\theta\in[0,2\pi]} \mu_\Delta(\mathscr{S}(e^{i\theta}I_n, M)) < 1$;
3. $\rho(M_{11}) < 1$ and $\max_{|\delta_1|\leq 1} \mu_\Delta(\mathscr{S}(\delta_1 I_n, M)) < 1$;
4. $\rho_{\Delta_P}(M) < 1$.

(1)⇔(2) is clear.
(2)⇔(3) follows from subharmonicity of the function μ_Δ (Lemma 3.7 says that is $\mu_\Delta(\cdot) = \max_{\triangle\in\mathbb{B}_\Delta} \rho(\triangle\cdot)$) and the maximum principe. The remaining equivalence is an immediate consequence of Main Loop Theorem.

Theorem (Main Loop Theorem)

$$\mu_\Delta(M) < 1 \Leftrightarrow \begin{cases} \mu_2(M_{22}) < 1, \\ \max_{\Delta_2 \in \mathbb{B}_2} \mu_1(\mathscr{S}(M, \Delta_2)). \end{cases}$$

Simillar results are possible when the upper bound is used instead of μ. For any $D \in \mathbb{D} \subset \mathbb{C}^{n\times n}$, where $\mathbb{D}$ is the scaling set for Δ, define

$$M_D = \begin{bmatrix} M_{11} & M_{12}D^{-1/2} \\ D^{1/2}M_{21} & D^{1/2}M_{22}D^{-1/2} \end{bmatrix}.$$

Moreover we need

$$\Delta_\sigma = \mathbb{C}^{m\times m},$$

$$\Delta_N = \left\{\text{diag}[\delta_1 I_n, \triangle_2] : \delta_1 \in \mathbb{C},\ \triangle_2 \in \Delta_\sigma\right\}.$$

Observe two important things. First that $\mu_\delta = \overline{\sigma}$, and the second that Δ_N is μ-simple (this is the content of Theorem 9.6).

The following are equivalent:

1. $\rho(M_{11}) < 1$ and $\inf_{D\in\mathbb{D}} \|D^{1/2}\mathcal{G}(M_D)D^{-1/2}\|_\infty < 1$,
2. $\rho(M_{11}) < 1$ and $\inf_{D\in\mathbb{D}} \max_{|\delta|\leq 1} \overline{\sigma}[D^{1/2}\mathscr{S}(\delta I_n, M_D)D^{-1/2}] < 1$,
3. $\rho(M_{11}) < 1$ and $\inf_{D\in\mathbb{D}} \max_{|\delta|\leq 1} \mu_{\Delta_\sigma}(\mathscr{S}(\delta I_n, M_D)) < 1$,
4. $\inf_{D\in\mathbb{D}} \mu_{\Delta_N}(M_D) < 1$,
5. $$\inf_{D\in\mathbb{D}, X\in\mathbb{C}^{n\times n}, X=\overline{X}^t>0} \overline{\sigma}\left(\begin{bmatrix} X^{1/2} & 0 \\ 0 & D^{1/2} \end{bmatrix} M \begin{bmatrix} X^{-1/2} & 0 \\ 0 & D^{-1/2} \end{bmatrix}\right) < 1,$$

where $\mathcal{G}(M) = M_{22} + M_{21}(I - M_{11})^{-1}M_{12}$.
Observe $D^{1/2}\mathcal{G}(M)D^{-1/2} = \mathcal{G}(M_D)$.

For (1)⇔(2) it is enough to remaind the definition of $\| \|_\infty$

$$\|G\|_\infty = \max_{|z|\geq 1} \overline{\sigma}(G(z))$$

(the definition is on the page 78 on the upper left). (2)⇔(3) follows from the previous observation and

$$D^{1/2}\mathscr{S}(\delta I_n, M_D)D^{-1/2} = \mathscr{S}(\delta I_n, M_D).$$

(3)⇔(4) is just the application of the Main Loop Theorem. To obtain (4)⇔(5) we need Theorem 8.4.

Theorem (8.4)

Suppose that Δ_N is μ-simple. Then for every $M \in \mathbb{C}^{(n+n)\times(n+m)}$,

$$\mu_{\Delta_N}(M) = \inf_{D\in\mathbb{D}} \overline{\sigma}(D^{1/2} M D^{-1/2}).$$

Now, after some computation

$$\mathrm{diag}[D_1^{1/2}, I_M] M_D \mathrm{diag}[D^{-1/2}, I_M] = \\ \mathrm{diag}[D_1^{1/2}, D^{1/2}]\, M\, \mathrm{diag}[D_1^{-1/2}, D^{-1/2}].$$

Let $M \in \mathbb{C}^{(n+n_p+m)\times(n+n_p+m)}$, partioned as below, relating several variable of a linear system by

$$\begin{bmatrix} x_{k+1} \\ e_k \\ z_k \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{12} \\ M_{13} \end{bmatrix}.$$

Let Δ be a prescribed $m \times m$ block structure.
As before we easily compute a linear fractional transformation $\mathscr{S}(M, \triangle)$ in this situation. Namely

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} + \begin{bmatrix} M_{13} \\ M_{23} \end{bmatrix} \triangle (I - M_{33}\triangle)^{-1} \begin{bmatrix} M_{31} & M_{32} \end{bmatrix}.$$

Put

$$\mathbf{\Delta}_N = \left\{ \mathrm{diag}[\delta_1 I_n, \triangle_2] : \delta_1 \in \mathbb{C},\ \triangle_2 \in \mathbb{C}^{n_p \times n_p} \right\},$$

$$\mathbf{\Delta}_S = \left\{ \mathrm{diag}[\triangle_N, \triangle] : \triangle_N \in \mathbf{\Delta}_N,\ \triangle \in \mathbf{\Delta} \right\},$$

$$\mathbf{\Delta}_P = \left\{ \mathrm{diag}[\triangle_2, \triangle] : \triangle_2 \in \mathbb{C}^{n_p \times n_p},\ \triangle \in \mathbf{\Delta} \right\}$$

Theorem (Time-invariant, robust performance)

Given the matrices and sets as defined above, the following conditions are equivalent:

1. $\mu_{\Delta_s}(M) < 1$,
2. $\mu_{\Delta}(M_{33}) < 1$ *and* $\max_{\triangle \in \mathbb{B}_{\Delta}}(\mathscr{S}(M, \triangle)) < 1$,
3. $\rho(M_{11}) < 1$ *and* $\max_{|\delta| \leq 1} \mu_{\Delta_P}(\mathscr{S}(\delta I_n, M) < 1$,
4. $\rho(M_{11}) < 1$ *and* $\max_{\theta \in [0,2\pi]}(\mathscr{S}(e^{i\theta}\delta I_n, M) < 1$.

(1)⇔(2) and (1)⇔(3) hold due to Main Loop Theorem applied to $\mathbb{C}^{(n+n_p)+m}$ and $\mathbb{C}^{n+(n_p+m)}$, respectively. (3)⇔(4) we have already proved (at the begining).