Maximum-modulus theorem

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$$\Longrightarrow \exists \widetilde{z}_1, \ldots, \widetilde{z}_n \text{ roots of } f + g \text{ such that } |\widetilde{z}_j - z_j| < \varepsilon.$$

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Theorem (6.5, maximum-modulus theorem, Packard-Balsamo)

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)},$$

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$$\max_{Q_2 \in \boldsymbol{Q}_2} \mu_1(\mathscr{S}(M, Q_2)) = \max_{\triangle_2 \in \boldsymbol{B}_2} \mu_1(\mathscr{S}(M, \triangle_2)).$$

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Remark (6.6)

Similarity of the maximum-modulus theorem to a result of Boyd-Desoer:

$$\max_{|z|=1} \mu_{\mathbf{\Delta}}(H(z)) = \max_{|z|\leq 1} \mu_{\mathbf{\Delta}}(H(z))$$

for $H \in \mathcal{O}(\mathbb{D}, \mathbb{C}^{n\times n}) \cap \mathcal{C}(\overline{\mathbb{D}})$.

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