1. Formulation and basic facts

Let $\mathbb{K}$ denote either $\mathbb{C}$ or $\mathbb{R}$ and let $F_j \in \mathbb{K}[X_1, ..., X_n]$, $j = 1, ..., n$. Put

$$\mathcal{P}(\mathbb{K}^n) := \{ F = (F_1, \ldots, F_n) : \mathbb{K}^n \to \mathbb{K}^n \},$$

$$\text{Jac} F(x) := \det \left[ \frac{\partial F_i}{\partial x_j}(x) : i, j = 1, \ldots, n \right]$$

and recall the formulation of the n-dimensional ($n \geq 2$) Jacobian Conjecture

$$(JC)_n \quad [F \in \mathcal{P}(\mathbb{K}^n), \, \text{Jac} F = \text{const} \neq 0] \Rightarrow [F \text{ is injective}].$$

and the so called Generalized Jacobian Conjecture, for short $(GJC)$, namely

$$(GJC) \quad (JC)_n \text{ holds for every } n \geq 2.$$

Note that if $\mathbb{K} = \mathbb{C}$, then, due to Osgood’s theorem, the condition $\text{Jac} F = \text{const} \neq 0$ is necessary for injectivity of $F$. In 1998 the Jacobian Conjecture was placed on Smale’s list (Mathematical Problems for the Next Century ([Sm])).

FACTS:

(i) It is known that any injective polynomial map of $\mathbb{K}^n$ is bijective ([BBR], [KR]) and, moreover, each injective polynomial map $F$ of $\mathbb{C}^n$ is a polynomial automorphism, i.e. the inverse $F^{-1}$ exists and is a polynomial mapping ([BCW], [W], [Y]).

(ii) If $F$ is a polynomial automorphism, then $\deg F^{-1} \leq (\deg F)^{n-1}$ ([BCW], [RW]).

(iii) Due to the Lefschetz Principle one can check that the formulation of the Jacobian Conjecture for the field $\mathbb{C}$ covers the case of the Jacobian Conjecture formulated for any field/domain of characteristic zero.
The Jacobian Conjecture and the Dixmier Conjecture

Definition 1.1. Let $k$ denote a field of characteristic 0. The $n$-th Weyl algebra over a field $k$ is the $k$-subalgebra $W_n = W_n(X_1, ..., X_n)$ of $k$-linear endomorphisms of $k[X_1, ..., X_n]$ generated by the multiplication maps $f$.

$$ f : k[X_1, ..., X_n] \ni g \rightarrow fg \in k[X_1, ..., X_n], \quad f \in k[X_1, ..., X_n] $$

and the $k$-derivations $\frac{\partial}{\partial X_i}$ on $k[X_1, ..., X_n]$, $i = 1, ..., n$.

We also write $W_n = k[X_1, ..., X_n, \partial_1, ..., \partial_n]$, where $\partial_i := \frac{\partial}{\partial X_i}$, $i = 1, ..., n$. One easily verifies the commutator relations

$$ [\partial_i, X_j] = \delta_{i,j}, \quad [\partial_i, \partial_j] = 0, \quad [X_i, X_j] = 0 \text{ for all } i, j. $$

Due to the above equations every element $P \in W_n$ can be written uniquely as a finite sum of the form

$$ P = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha $$

where $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $a_\alpha = a_{\alpha_1, ..., \alpha_n}(X_1, ..., X_n) \in k[X_1, ..., X_n]$, $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We have the following.

Proposition 1.2. Any non-zero $k$-endomorphism of $W_n$ is injective.

Proof. If $h$ be a nontrivial $k$-endomorphism of $W_n$, then $\ker h \neq W_n$. Since $W_n$ is a simple ring (i.e. every two-sided ideal in $W_n$ is either zero or the whole ring) it follows that $\ker h = 0$, so every $k$-endomorphism of $W_n$ is injective.

The question what about epimorphism is the subject of the so called Dixmier Conjecture (for short (DC)).

(DC)$_n$ Every $k$-endomorphism of $W_n$ is an epimorphism (i.e. is an isomorphism).

In fact only (DC)$_1$ was formulated by Dixmier ([Di]) and (DC)$_n$ is still unsolved for any $n \geq 1$. We recall the relations between the Jacobian Conjecture and the Dixmier Conjecture.

(i) (DC)$_n$ implies (JC)$_n$ (book fact).

(ii) (JC)$_{2n} \Rightarrow$ (DC)$_n$ (June 2005 - Y. Tsuchimoto [T], Dec. 2005 (arXiv) - A. Belov and M. Kontsevich [BK]).

Hence we have the following equivalence.

Theorem 1.3. ([T], [BK]). (GJC) is equivalent to the Generalized Dixmier Conjecture, i.e. (DC)$_n$ holds for every $n \geq 1$. 2
Let us go back to (JC). If $\mathbb{K} = \mathbb{R}$ and $\text{Jac} F(x) > 0$ for any $x \in \mathbb{R}^n$, then we can ask about injectivity of $F$ and we have the so-called Strong Real Jacobian Problem. In 1994, Pinchuk ([Pi]) gave an example showing that the Strong Real Jacobian Problem is false even in the case of $\mathbb{R}^2$, so also in $\mathbb{R}^n$ for any $n \geq 2$. Note that up to this time the Jacobian Conjecture remains unsolved even if $n = 2$.

Since $F \in \mathcal{P}(\mathbb{C}^n)$ can be treated as $\hat{F} = (\text{Re } F, \text{Im } F) \in \mathcal{P}(\mathbb{R}^{2n})$ and $\text{Jac} \hat{F}(x, y) = |\text{Jac } F(x + iy)|^2$, it is evident that $(JC)^2_n$ for $\mathbb{R}[X_1, \ldots, X_{2n}] \Rightarrow (JC)_n$ for $\mathbb{C}[X_1, \ldots, X_n]$, so ”the real (GJC)” implies ”the complex (GJC)”. But we even do not know if the following (”little (JC)”) is true.

**Problem 1.4.** real $(JC)_n$ $\implies$ complex $(JC)_n$, more precisely $[(F \in \mathcal{P}(\mathbb{R}^n), \text{Jac } F = 1) \Rightarrow F$ - injective on $\mathbb{R}^n] \Rightarrow [F$ - injective on $\mathbb{C}^n]$

We have the following.

**Proposition 1.5.** Let $F = (F_1, \ldots, F_n) \in \mathcal{P}(\mathbb{K}^n)$ be a quadratic map, i.e. $\deg F := \max\{\deg F_j : j = 1, \ldots, n\} \leq 2$. Then $F$ is injective if and only if $\text{Jac } F(x) \neq 0$ for any $x \in \mathbb{K}^n$.

**Proof.** It follows immediately from the following identity

$$F(x) - F(y) = F'(\frac{x + y}{2})(x - y) \quad \text{for every } x, y \in \mathbb{K}^n. \quad \square$$

Note that it is sufficient to consider only polynomial mappings of the form $F(x) = x + g(x)$, where no constant or linear term is included in $g$.

**Problem 1.6.** Let $V$ denote a (normed or unitary) vector space over $\mathbb{K}$, let $g : V \times V \to V$ be a (continuous) symmetric bilinear mapping such that there exists $p \in \mathbb{N}$ such that $g(x, \cdot)^p = 0 \in L(V, V)$ for any $x \in V$. Is the mapping $F(x) := x - g(x, x) : V \to V$ surjective?

**Remark 1.7.** Since $\delta F(x) = I - 2g(x, \cdot)$ is a linear isomorphism, $F$ is injective (but injectivity does not imply surjectivity even in the case of continuous linear maps of $l^2$). If $p=2$, then the answer is positive (for any $k$-linear $g$) and $F^{-1}(x) = x + g(x, \ldots, x)$.

2. **Reduction of the degree**

Put $\deg F := \max\{\deg F_j : j = 1, \ldots, n\}$ if $F = (F_1, \ldots, F_n)$. By the procedure of adding new variables and composing with polynomial automorphisms or quasi-dilations (a stabilization method) we obtain simplified class of investigated polynomials in (GJC).
Theorem 2.1 ([Y, BCW, D1]). It is sufficient to consider in (GJC) only polynomial mappings of the so called cubic homogeneous form \( F = I + H \), where \( H = (H_1, \ldots, H_n) \) and \( H_j : \mathbb{K}^n \to \mathbb{K} \) is a cubic homogeneous polynomial of degree 3 or \( H_j = 0, \ j = 1, \ldots, n \ (n > 1). \)

As a consequence of the above proposition and the mentioned Pinchuk’s example ([Pi]) we get

Corollary 2.2. It is impossible to reduce (by the stabilization method) any polynomial mapping \( F \in \mathcal{P}(\mathbb{K}^n) \) to a quadratic map \( \hat{F} \in \mathcal{P}(\mathbb{K}^N) \) \( (N \geq n) \) preserving injectivity and nowhere vanishing jacobian.

It is easy to prove the following:

Proposition 2.3 ([BCW, D1]). Let \( F = I + H \) have a cubic homogeneous form. Then

\[
\text{Jac } F = 1 \iff H'(x) \text{ is a nilpotent matrix for any } x \in \mathbb{K}^n.
\]

Note that \( H'(x) = 3\tilde{H}(x, x, \cdot) \), where \( \tilde{H} \) denotes the unique symmetric three-linear mapping such that \( \tilde{H}(x, x, x) = H(x) \). Hence, if \( \text{Jac } (I + H) = 1 \), then by Proposition 2.3 the matrix

\[
H_x := \tilde{H}(x, x, \cdot) = \frac{1}{3}H'(x)
\]

is nilpotent.

Therefore, for every \( x \in \mathbb{K}^n \), there exists the index of nilpotency of the matrix \( H_x \), i.e. there exists a natural number \( p(x) \) such that \( H_x^{p(x)} = 0 \) and \( H_x^{p(x)-1} \neq 0 \). It is evident, that \( 1 \leq p(x) \leq 1 + \text{rank } H_x \leq n \) for every \( x \in \mathbb{K}^n \). We define the index of nilpotency of the mapping \( F = I + H \) to be the number

\[
\text{ind } F := \sup \{ p(x) \in \mathbb{N} : H_x^{p(x)} = 0, \ H_x^{p(x)-1} \neq 0, \ x \in \mathbb{K}^n \}.
\]

If \( v = (v_1, \ldots, v_n)^T \) is a column vector and \( k \in \mathbb{N} \), then we denote by \( v^{\circ k} \) the \( k-th \) Hadamard-Schur power of \( v \), i.e. \( v^{\circ k} := ((v_1)^k, \ldots, (v_n)^k)^T \) and by \( \Delta(v^{\circ k}) \) we denote the diagonal \( n \times n \) matrix

\[
\Delta(v^{\circ k}) := \begin{pmatrix}
(v_1)^k & 0 & 0 & \cdots & 0 & 0 \\
0 & (v_2)^k & 0 & \cdots & 0 & 0 \\
& \cdots & \vdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & (v_{n-1})^k & 0 \\
0 & 0 & \cdots & 0 & 0 & (v_n)^k
\end{pmatrix}.
\]

Now we present a theorem which allow us to reduce the verification of the Generalized Jacobian Conjecture to the investigation of polynomial mappings of the so called cubic linear form.
Theorem 2.4 (cubic linear formulation of (GJC), [D1, D2, D3]). (i) In order to verify (GJC) it is sufficient to check it only for polynomial mappings $F = (F_1, ..., F_n)$ of the cubic linear form (CLF), i. e.

$$F(x_1, ..., x_n) = \begin{pmatrix} x_1 + (a_1^1 x_1 + \ldots + a_1^n x_n)^3 \\ x_2 + (a_2^1 x_1 + \ldots + a_2^n x_n)^3 \\ \vdots \\ x_n + (a_n^1 x_1 + \ldots + a_n^n x_n)^3 \end{pmatrix} = \begin{pmatrix} x_1 + (a_1 x)^3 \\ x_2 + (a_2 x)^3 \\ \vdots \\ x_n + (a_n x)^3 \end{pmatrix},$$

briefly

$$F(x) = x + (Ax)^3,$$

where $x = (x_1, ..., x_n) \in \mathbb{K}^n$, $a_j x := a_j^1 x_1 + ... + a_j^n x_n$, $A := [a_i^j : i, j = 1, ..., n]$ and $(Ax) := (a_1 x, ..., a_n x)^T$ is one column matrix/vector.

(ii) Without loss of generality we can additionally assume in (i) that $A$ has an additional nilpotent property, namely there exists $c \in \mathbb{K}^n$ such that

$$A = (Ax)^3 \bigg|_{x=c} = 3\Delta((Ac)^2) A = \begin{bmatrix} 3(a_1 c)^2 a_1^1 & \ldots & 3(a_1 c)^2 a_1^n \\ \vdots & \ddots & \vdots \\ 3(a_n c)^2 a_n^1 & \ldots & 3(a_n c)^2 a_n^n \end{bmatrix}$$

and $\text{ind} A = \text{ind} F$.

(iii) Without loss of generality we can additionally assume in (i) that $A^2 = 0$.

Remark 2.5. (i) Note that if you take $A$ in (i) such that $A^2 = 0$, then usually $2 = \text{ind} A < \text{ind} F$.

(ii) The cubic homogeneous form (Yagzev’s form) is invariant under the action of the full linear group $GL_n(\mathbb{K})$, i. e. if $F$ has Yagzev’s form and $L \in GL_n(\mathbb{K})$, then $L \circ F \circ L^{-1}$ has also Yagzev’s form. The cubic linear form is not invariant under the action of the full linear group $GL_n(\mathbb{K})$, it is invariant under the action of the subgroup generated by all permutations and dilations $J(x_1, ..., x_n) = (c_1 x_1, ..., c_n x_n)$, $c_1 \cdot ... \cdot c_n \neq 0$.

Up to now we even do not know whether polynomials $F_j(x) = x_j + H_j(x)$, $j = 1, ..., n$, are irreducible when $\text{Jac} F = 1$ although it is necessary if $F$ is a polynomial automorphism; the cubic linear form is nicer, namely

Remark 2.6. If $F = (F_1, ..., F_n)$ is of (CLF) and $\text{Jac} F = 1$, then the polynomials $F_j(x) + c = x_j + (a_j x)^3 + c$, $c \in \mathbb{K}$, $j = 1, ..., n$, are irreducible.
Recall that a mapping $F$ is a tame automorphism if the mapping $F - F(0)$ is a finite composition of linear automorphisms and shears (called also elementary automorphisms). A shear is an elementary triangular transformation of the form

$$T(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i + f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), x_{i+1}, \ldots, x_n).$$

We present a theorem which summarizes some partial results on (GJC) formulated in the cubic linear form.

**Theorem 2.7.** (D1, D2, DR]). For arbitrary $n > 1$ the following holds:

If a polynomial map $F = (F_1, \ldots, F_n): \mathbb{K}^n \to \mathbb{K}^n$ with $\text{Jac} F = 1$ has a cubic linear form and if

$$\text{rank} A < 3 \quad \text{or} \quad \text{corank} A < 3 \quad \text{or} \quad \text{ind} F = 1, 2, 3, n,$$

then $F$ is a tame polynomial automorphism.

**Corollary 2.8.** Combining th.2.7 with Hubbers’ results (cf. [E]) we obtain the following. If $F(x) = x + (Ax)^3 : \mathbb{K}^n \to \mathbb{K}^n$ and $\text{Jac} F = 1$, then $F$ is an automorphism provided $n < 8$.

3. Symmetric reduction of (GJC)

We show that it is possible to reduce (GJC) to complex cubic homogeneous Yagzev’s form with the COMPLEX symmetric jacobian matrix $F'(x)$ for any $x \in \mathbb{C}^n$, if $x \in \mathbb{C}^n$ even if we start from $F: \mathbb{R}^n \to \mathbb{R}^n$.

**Theorem 3.1.** (i) It is sufficient to verify (GJC) only for polynomial mappings of a real symmetric cubic homogeneous form

$$F(x, y) = (-x, y) + S(x, y) : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \quad \text{(for every } n > 1),$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$, $S = (S_1, \ldots, S_{2n}) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a cubic homogeneous polynomial mapping of degree 3 and, for any $(x, y) \in \mathbb{R}^{2n}$, $S'(x, y)$ is a real symmetric matrix.

(ii) It is sufficient to verify (GJC) only for polynomial mappings of a complex symmetric cubic homogeneous form

$$F(x) = x + \tilde{S}(x) : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \quad \text{(for every } n > 1),$$

where $\tilde{S} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a cubic homogeneous polynomial mapping of degree 3 and, for any $x \in \mathbb{C}^{2n}$, $\tilde{S}'(x)$ is a complex symmetric nilpotent matrix.

**Proof.** Let $M_{p,q}(\mathbb{K})$ denote the set of $p \times q$ matrices with coefficients in $\mathbb{K}$, $M_p(\mathbb{K}) := M_{p,p}(\mathbb{K})$. Note that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{p+r,p+r}(\mathbb{K})$ is a symmetric matrix if and only if $a = a^T \in M_{p,p}$, $c = b^T \in M_{r,p}$, $d = d^T \in M_{r,r}$. Let $x = (x_1, \ldots, x_n)^T$, $v = (v_1, \ldots, v_n)^T$.
and let $F = (F_1, ..., F_n)^T$ be a polynomial mapping of \( \mathbb{K}^n \) (the transpose \( T \) indicates that we treat \( x, v, F \) as a column vector/matrix). If \( Q := Q^{[0]} + Q^{[1]} + ... + Q^{[d]} : \mathbb{K}^n \to \mathbb{K} \) is a non-zero polynomial with homogeneous terms \( Q^{[j]} \) of degree \( j \), then we put \( \operatorname{ad} Q := \min\{ j : Q^{[j]} \neq 0, j = 0, ..., d \} \). It is obvious that without loss of generality we can consider \( F \) having the form

$$F(x) = (x_1 + Q_1(x), ..., x_n + Q_n(x))^T : \mathbb{K}^n \to \mathbb{K}^n,$$

where \( Q_j \) is a polynomial of \( \operatorname{ord} Q_j \geq 2, j = 1, ..., n \).

Take

$$g(v, x) := v_1 F_1(x) + ... + v_n F_n(x)$$

and define \( G \in \mathcal{P}(\mathbb{K}^{2n}) \) by the formula

$$G(v, x) := \nabla g(v, x)^T = (\frac{\partial g}{\partial v_1}, ..., \frac{\partial g}{\partial v_n}, \frac{\partial g}{\partial x_1}, ..., \frac{\partial g}{\partial x_n})^T.$$

One can easily verify that

$$G(v, x) = (F_1(x), ..., F_n(x), \sum_{k=1}^n v_k \frac{\partial F_k}{\partial x_1}, ..., \sum_{k=1}^n v_k \frac{\partial F_k}{\partial x_n})^T$$

$$= (F(x), [F'(x)]^T v).$$

Obviously, \( G \) is injective if and only if \( F \) is. We calculate

$$G'(v, x) = \begin{bmatrix} 0 & F'(x) \\ F'(x)^T & [F''(x)]^T (v, \cdot) \end{bmatrix},$$

where

$$[F''(x)]^T (v, \cdot) = \begin{bmatrix} \sum_{k=1}^n v_k \frac{\partial^2 F_k}{\partial x_1^2} & ... & \sum_{k=1}^n v_k \frac{\partial^2 F_k}{\partial x_1 \partial x_n} \\ ... & ... & ... \\ \sum_{k=1}^n v_k \frac{\partial^2 F_k}{\partial x_n \partial x_1} & ... & \sum_{k=1}^n v_k \frac{\partial^2 F_k}{\partial x_n^2} \end{bmatrix}$$

is symmetric \( n \times n \) matrix since \( \frac{\partial^2 F_k}{\partial x_i \partial x_j} = \frac{\partial^2 F_k}{\partial x_j \partial x_i} \). Hence \( G'(v, x) \) is a symmetric \( 2n \times 2n \) matrix and

$$G'(0, 0) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

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where $I$ denotes the $n \times n$ identity matrix. Thus taking the orthogonal matrix
\[
M = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}
\]
we get
\[
M^T \circ G'(0,0) \circ M = E,
\]
where
\[
E = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.
\]
Put $w = (v,x) \in \mathbb{K}^{2n}$ and $P(w) := M^T \circ G \circ M(w) = E(w) + S(w) : \mathbb{K}^{2n} \to \mathbb{K}^{2n}$. Then the map $P = E + S$ is polynomial and, for any $w \in \mathbb{K}^{2n}$, $S'(w)$ is a symmetric matrix. Note that by the Laplace theorem $\text{Jac} P = \text{Jac} G = (-1)^n(\text{Jac} F)^2$ and $F$ is injective if and only if $P$ is injective.

Ad (i). Obviously, if $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a cubic homogenous mapping, then $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is also a cubic homogenous mapping and, for any $w \in \mathbb{R}^{2n}$, $S'(w)$ is a real symmetric matrix.

Ad (ii). Now it is evident that if we want to get (by similarity) the identity matrix instead of $E$ and to preserve a symmetric form of the matrix $S'$ we have to use some complex matrix, e.g. the complex dilation
\[
J = \begin{bmatrix} iI & 0 \\ 0 & I \end{bmatrix}.
\]
Put $\hat{P}(z) := J \circ P \circ J(z) = x + \hat{S}(z)$, where $\hat{S} = J \circ S \circ J : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Evidently, if $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a cubic homogenous mapping and, for any $w \in \mathbb{R}^{2n}$, $S'(w)$ is a real symmetric matrix, then $\hat{S} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a cubic homogenous mapping and $\hat{S}'(z)$ is a complex symmetric matrix for any $z \in \mathbb{C}^{2n}$. Evidently, $F$ is injective if and only if $\hat{P}$ is injective. Since $\text{Jac} \hat{P} = (i)^{2n}(-1)^n(\text{Jac} F)^2 = (\text{Jac} F)^2 = 1$, for any $z \in \mathbb{C}^{2n}$ the matrix $\hat{S}'(z)$ is a complex symmetric matrix which is also nilpotent. □

**Remark 3.2.** The trick with a polynomial $g(v,x)$ (which is probably a folk result) we adopted from [M] where the sketch of the reduction (JC)$_n$ to the complex symmetric case of (JC)$_{2n}$ is given. The reduction of (GJC) to the complex symmetric case has also been proved in [BE].

We know by th. 2.4 and prop. 2.3 that it is sufficient to verify (GJC) only for polynomial mappings of the cubic linear form $F(x) = x + (Ax)^{o,3}$ with nilpotent matrix $[(Ax)^{o,3}]' = 3\Delta((Ax)^{o,2})A$ for any $x \in \mathbb{K}^n$. Although at present we cannot reduce every polynomial mapping to the cubic linear form $F(x) = x + (Ax)^{o,3}$ with complex symmetric nilpotent
matrix \([(Ax)^3]\) it is natural, in view of th. 3.1, to ask if the Jacobian Conjecture is true in this particular case. We have prove the following.

**Theorem 3.3.** If \(F(x) = x + (Ax)^3 : \mathbb{C}^n \to \mathbb{C}^n\) is of the cubic linear form and the differential \([(Ax)^3]\) is a complex symmetric nilpotent matrix for any \(x \in \mathbb{C}^n\), then \(F\) is a tame automorphism.

(We recall that the nilpotency of \([F(x) - x]\) is equivalent to the condition \(\text{Jac} F = 1\).)

**Proof.** (0) Notice that if \(P\) is a permutation (i.e. the matrix of a permutation) and \(F\) has the cubic linear form, then the mapping \(P^{-1} \circ F \circ P\) has also (CLF). Observe that symmetry and nilpotency of \([(Ax)^3]\) is preserved since \(P^{-1} = P^T\) and hence the matrix \([P^T \circ (A(Px))^3] = P^T \circ [(Ay)^3] \circ P\), where \(y = P(x)\), is symmetric and nilpotent. Therefore, after permutation similarity \(P^{-1} \circ F \circ P\) of the mapping \(F\) (which is equivalent to taking \(P^{-1} \circ A \circ P\) instead of \(A\)) we preserve the cubic linear form of \(F\) as well as symmetry and nilpotency of the "new" matrix \(A\).

**Induction.** One can check the theorem by simple calculations if \(n = 2\).

**I.** If \(i\)-th row/column of \(A\) is equal to 0, then - by the symmetry - \(i\)-th column/row of \(A\) is also equal to 0. Due to (0) we can assume that it is the last column and then conclude that \(F\) is injective if and only if the mapping \(G(x_1, ..., x_{n-1}) := (F_1(x_1, ..., x_{n-1}, 0), ..., F_{n-1}(x_1, ..., x_{n-1}, 0)) : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}\) is symmetric differential is injective. Obviously, \(F = (G, x_n) \circ R\), where \(R(x) = (x_1, ..., x_{n-1}, x_n + F_n(x_1, ..., x_{n-1}))\). By induction \(G\) is a tame automorphism and so is \(F\).

**II.** Assume that no column/row of \(A := [a_{ij} : i, j = 1, ..., n]\) is equal to 0.

(i) Put \(N_i = \#\{j : j \in \{1, ..., i-1, i+1, ..., n\}, a_{ij} \neq 0\}\) and \(N := \max\{N_i : i = 1, ..., n\}\). If \(q = 1\) we do nothing. If \(q > 1\) we change \(q\)-th row with \(1\)-st row and then permute \(q\)-th column with \(1\)-st column.

(ii) Now we have in the first row of the "new" matrix \(A\) exactly \(N\) coefficients different from 0 (we do not count \(a_{11}\)), say \(a_{1j_1}^{j_1}, ..., a_{1j_N}^{j_N} \neq 0\) where \(j_1 > 1\). Permute the columns \(j_1, ..., j_N\) into the columns \(2, 3, ..., N + 1\) and the rows in the same way.

(iii) If all columns \(a_{N+2}^N, ..., a_n^N\) in submatrix \(B := [a_{ij} : i = 1, ..., N + 1, j = 1, ..., n]\) are equal to 0, then also by symmetry \(a_{ij}^i = 0\) whence \(i =
$N + 2, \ldots, n$, $j = 1, \ldots, N + 1$ and we can represent $F$ as the composition of two mappings

$$Q(x) = (F_1(x_1, \ldots, x_{N+1}), \ldots, F_{N+1}(x_1, \ldots, x_{N+1}), x_{N+2}, \ldots, x_n)$$

and

$$R(x) = (x_1, \ldots, x_{N+1}, F_{N+2}(x_{N+2}, \ldots, x_n), \ldots, F_n(x_{N+2}, \ldots, x_n),),$$

i.e. $F = Q \circ R$. By induction $Q$ and $R$ are tame automorphisms, so is $F$.

(iv) If $s$ columns of the columns $a_1^{N+2}, \ldots, a_n^N$ in submatrix $B := [a_i^j : i = 1, \ldots, N + 1, j = 1, \ldots, n]$ are not equal to 0, $0 < s \leq n - N - 1$, then we permute them in such a way that only $k$ last columns in $B$ are equal to 0, $k := n - N - 1 - s \geq 0$. Afterwards we permute the appropriate rows and the ”new” matrix $[(Ax)^{o3}]$ is symmetric and nilpotent for any $x \in \mathbb{C}^n$.

(v) If $k > 0$, then the argument analogous to that given in (iii) finishes the proof.

(vi) If $k = 0$, i.e. all columns $a_1^{N+2}, \ldots, a_n^N$ in submatrix $B := [a_i^j : i = 1, \ldots, N + 1, j = 1, \ldots, n]$ are different from 0, then it is easy to check that all rows of the symmetric matrix $[(Ax)^{o3}] = 3\Delta((Ax)^{o2})A$ are different from 0 and parallel to the row $(a_1 x)^2 a_1$. Hence rank $A = 1$ and by th. 2.7 the mapping $F$ is a tame automorphism. □

Remark 3.4. The more refined version of the above theorem was proved later in [E1]. Let $\langle v, x \rangle := v_1 x_1 + \ldots + v_n x_n, f := (v, x)^{d+1}$ and $\nabla f := \frac{\partial f}{\partial x_1} + \ldots + \frac{\partial f}{\partial x_n}$. A polynomial mapping of the form $x + \nabla f$ with $\langle v, v \rangle = 0$ is called a D-mapping of type 1. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be two sets of independent variables. A polynomial map $F : k^{n+m} \to k^{n+m}$ admits a separation in the variables $x$ and $y$ if there exist polynomial mappings $G, H : k^{n+m} \to k^{n+m}$ of the form $G = (G_1(x), \ldots, G_m(x), y)$ and $H = (x, H_1(y), \ldots, H_m(y))$ such that $F = G \circ H = H \circ G$ (we write $F = G(x) \circ H(y)$). In a similar way one can define when a polynomial map $F : k^{n_1 + \ldots + n_s} \to k^{n_1 + \ldots + n_s}$ admits a separation in a finite set of variables $x^{(1)} = (x_1^{(1)}, \ldots, x_{n_1}^{(1)}), \ldots, x^{(s)} = (x_1^{(s)}, \ldots, x_{n_s}^{(s)})$.

Theorem ([E1]). Let $F = x + (Ax)^{od}$ be a D-mapping of degree $\geq 2$. If $JF$ is invertible and symmetric, then there exists a permutation map $P$ such that $P^{-1} \circ F \circ P$ admits a separation in a finite set of variables, say $F^{(1)}(x^{(1)}) \circ \ldots \circ F^{(s)}(x^{(s)})$, where each $F^{(i)}$ is a D-mapping of type 1. In particular the Jacobian Conjecture holds for D-mappings with symmetric Jacobian matrices.
The next contributions towards (JC) would be the proof of the following.

**Theorem 3.5.** Let \( F(x) = x + (Ax)^3 : \mathbb{C}^n \to \mathbb{C}^n \) be of the cubic linear form, \( \text{Jac} \, F = 1 \) and let the matrix \( A \) be a complex symmetric nilpotent matrix. Then \( F \) is a tame automorphism.

**References**


