A symmetry of maps implies its chaos, i.e. gives $\infty$ many periodic points

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1. Abstract

2. Problems

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6. Nielsen Theory

7. Per. pts of self-map of the orbit space

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9. Bibliography
\[ f : X \rightarrow X, \] here \( X \) is a closed manifold of dim \( d \).

**Notation:**

\[ P^n(f) := \text{Fix}(f^n) \] the set of points of period \( n \),

\[ P_n(f) := P^n(f) \setminus \bigcup_{k|n<n} P^k(f) \] the set of points for which \( n \) is the **minimal period**, called shortly \( n \)-**periodic** points.

\[ P(f) := \bigcup_{1}^{\infty} P^n(f) = \bigcup_{1}^{\infty} P_n(f) \] the set of all **periodic points**.

\[ \text{Per}(f) := \{ n \in \mathbb{N} : P_n(f) \neq \emptyset \} \] the set of **minimal periods** of \( f \).
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Theorem (Shub & Sullivan 1974)

\[ f : X \rightarrow X \text{ such that } \{L(f^n)\}_{1}^{\infty} \text{ is unbounded} \]

If \( f \in C^1 \) then \( \#P(f) = \infty \)

Lefschetz-Hopf formula

\[ L(f^n) = \text{Ind}(f^n, X) = \sum_{x \in \text{Fix}(f^n)} \text{Ind}(f^n, x) \]

Main step: \( f \in C^1, f(0) = 0 \implies \{\text{Ind}(f^n, 0)\} \text{ is bounded} \)

provided it is defined.

Chow & Mallet-Paret & Yorke (81): the sequence is periodic of a period \( k \) defined by \( \sigma(Df(0)) \).
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**Conjecture (Shub 1974)**

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\limsup_n \sqrt{\# P^n(f)} \geq \limsup_n \sqrt{|L(f^n)|} = \rho_{es} > 1 \text{ if } \{L(f^n)\} \text{ is unb.}
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The rate of growth is at least exponential.

**Theorem (Babenko & Bogatyi 1991)**

Let \( f : X \rightarrow X, \ d = \dim X, \) and \( f \in C^1. \) Assume: \( \{L(f^n)\}_{1}^{\infty} \) is unb. Then there exists \( n_0 = n_0(f) \) such that \( \forall \ n \geq n_0 \)

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\# \text{Or}(f, n) \geq \frac{n - n_0}{D^{2[d+1/2]}},
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where \( D := \dim H^*(X; \mathbb{R}) \)

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Theorem ([JJWM1])

Let $g : S^d \to S^d$, $d \geq 1$, be a free homeomorphism of finite order $m > 1$, and $f : S^d \to S^d$ be a map that commutes with $g$. Suppose that $\deg(f) \notin \{-1, 0, 1\}$. Then for $\forall \ k \in \mathbb{N}$ we have

$$\#\text{Fix}(f^{km}) \geq m^2 k'$$

$k'$ is as in Definition [19]. In particular, for $k = m^s$ we have

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For $S^d$, $d \geq 1$, $\{L(f^k)\}_{1}^{\infty}$ is unbounded iff $\deg(f) \neq 0, \pm 1$.

Corollary ([JJWM1])

Let $f : S^d \to S^d$ be a continuous map with $\{L(f^n)\}_{1}^{\infty}$ unbounded. If $f$ commutes with a free homeomorphism $g : S^d \to S^d$ of order $m > 1$, then the set $\text{Per}(f)$ of minimal periods of $f$ and consequently the set $P(f)$ of periodic points are infinite.
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Eventually generalize for $G$-equivariant maps?
for $G$ not cyclic, or infinite compact

1. Few class of groups: There exists a classification of finite
groups which can act free (act smoothly) on the spheres!!,
e.g. they do not contain $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

2. The only infinite groups acting free on $S^d$:
$G = S^1$, $N(S^1) \subset S^3$, $S^3$.

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infinite compact, then $L(f) = 0$, consequently $L(f^k) = 0$ for
$\forall \ k$.

4. For the problem of existence of infinitely many periodic points,
one can always restrict the action to a cyclic subgroup of $G$.

**Remark**

Note that in the simplest case $G = \mathbb{Z}_2$, or $G = \mathbb{Z}_m$ the orbit
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The idea of proof

- The general idea of the proofs of the stated Theorems is to study a map $\bar{f} : M \to M$ of the quotient space $M := S^d / \mathbb{Z}_m$ induced by the $\mathbb{Z}_m$-equivariant map $f : S^d \to S^d$ in the problem.

- Next we estimate the number of periodic points of $\bar{f}$, and we ”lift” them to periodic points of $f$.

- To study periodic points of the induced map $\bar{f}$ we use the Nielsen theory adapted to this situation.

- It is worth pointing out that a direct application of the Nielsen number of iterations is inefficient since

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Below we include some facts about equivariant maps

**Proposition (Borsuk-Ulam)**

Suppose that $\mathbb{Z}_m$ acts freely on $S^d$, $d \geq 1$. If $f : S^d \to S^d$ is an equivariant map, then $\deg(f) \equiv 1 \mod m$.

For $m = 2$, this is the classical Borsuk-Ulam theorem which states that an odd map has odd degree.

$f, h : S^d \to S^d$ are homotopic $\iff$ $\deg f = \deg h$.

**Theorem (C, Bowszyc, R. Rubinsztein, [Rub].)**

Suppose that a finite group $G$ acts freely on $S^d$, $d > 1$. Then the natural map $[S^d, S^d]_G \to [S^d, S^d]$ is an injection, i.e. if two equivariant mappings have the same degree, then they are equivariantly homotopic.

Moreover the image of $[S^d, S^d]_G$ in $[S^d, S^d] = \mathbb{Z}$ is equal to $\{m\mathbb{Z} + 1\}$. 
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Comments

Note that taking the suspension of the map $f : S^1 \rightarrow S^1$, $f(z) = z^r$, $|r| \geq 2$ (with $\infty$ many periodic points) we get a map $\Sigma f$ of $S^2$ with the same dynamics as $z^r$.

On the other hand, the Shub example gives a map of $S^2$ which is a small perturbation of $\Sigma f$ but has only two non-wandering points.

(Note that the Shub example is not $\mathbb{Z}_2$-equivariant)

The stated Theorems say that any small equivariant perturbation, or more generally any equivariant continuous deformation of $f$ must possess infinitely many periodic points.
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In this section we denote by $p : \tilde{X} \rightarrow X$ a universal covering of $X$. For a $G$-space $X$ with a free action of a finite group $G$ and a map $X \rightarrow \tilde{X} = X/G$ onto the orbit space we will denote the covering $p : X \rightarrow \tilde{X}$, opposite to the notation used here.

Let $p : \tilde{X} \rightarrow X$ be a universal covering of a polyhedron.

$$O_X := \{ \alpha : \tilde{X} \rightarrow \tilde{X} : p\alpha = p \}$$

is the group of *deck transformations* of this covering.

Let $f : X \rightarrow X$ be a map and let $\text{lift}(f) = \{ \tilde{f} : \tilde{X} \rightarrow \tilde{X} : p\tilde{f} = fp \}$ denote the set of all lifts of $f$.

If we fix a lift $\tilde{f}_0$, then each other lift of $f$ can be uniquely written as $\alpha\tilde{f}_0$, $\alpha \in O_X$. 
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In [Jia1] Boju Jiang introduced a number $NF_k(f)$ which is a homotopy invariant and is the lower bound for $\#\text{Fix}(f^k)$.

**Theorem**

For any self-map $f : X \to X$ of a finite polyhedron and a fixed natural number $k \in \mathbb{N}$

$$\#\text{Fix}(f^k) \geq \sum_{r|k} (\#\text{IEOR}(f^r)) \cdot r$$

where $\text{IEOR}(f^r)$ denotes the set of irreducible ($\mathcal{I}$) essential ($\mathcal{E}$) orbits ($\mathcal{O}$) of Reidemeister ($\mathcal{R}$) classes of the map $f^r$. 
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Lemma

Consider the commutative diagram

\[ \tilde{Y} \xrightarrow{\tilde{f}} \tilde{Y} \]
\[ \downarrow p \quad \downarrow p \]
\[ Y \xrightarrow{f} Y \]

where \( p : \tilde{Y} \rightarrow Y \) is a finite regular covering of a fin. polyh. \( Y \).

Then

\[ \text{ind}(\tilde{f}) = r \cdot \text{ind}(f; p(\text{Fix}(\tilde{f}))) \]

where \( r = \#\{ \alpha \in \mathcal{O}_Y; \tilde{f}\alpha = \alpha\tilde{f} \} \). In particular
\[ \text{ind}(f; p(\text{Fix}(\tilde{f}))) \neq 0 \text{ if and only if } L(\tilde{f}) = \text{ind}(\tilde{f}) \neq 0. \]

\( \mathcal{O}_Y \) denotes the group of covering transformations of the regular covering \( p \); exceptionally in this Lemma we do not need to assume that the covering \( p \) is universal.

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Corollary

Let $\tilde{f} : M \to M$ be the map induced by an equivariant map $f : S^d \to S^d$ of degree $\neq 0, \pm 1$. Then all the Reidemeister classes of $f$ and of all its iterations are essential.

It is a key point which uses the fact that $M = S^d / G$. In general we need an information that

$$L(f) \neq 0 \implies \forall g \in G, g \neq e, \text{ we have } L(gf) \neq 0$$

Or in a weaker form:

For $\infty$ many $k$ (of a form which is related to $m = |G|$)

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Lemma

The Reidemeister relation of the map $\tilde{f} : \tilde{X} \to \tilde{X}$ induced by an equivariant map $f : X \to X$ is trivial. Thus $R(\tilde{f}) = O_{\tilde{X}} = \mathbb{Z}_m$.

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If $\tilde{f} : M \to M$ ($M = S^d/\mathbb{Z}_m$) is a map induced by an equivariant map $f : S^d \to S^d$, then we have

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Thus it remains to find the number of irreducible classes in $\mathcal{R}(\bar{f}^r)$ for such $r$ (or for every $r$). Let us recall that the class $A \in \mathcal{R}(\bar{f}^k)$ is reducible iff it belongs to the image of the map $i_{kl} : \mathcal{R}(\bar{f}^l) \rightarrow \mathcal{R}(\bar{f}^k)$ for an $l \mid k, l < k$. 
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Equivariant Nielsen theory for a free action on arbitrary $M$

**Theorem ([JJWM2])**

Let $M$ be a finite polyhedron with a free action of a finite group $G$ and $f : M^G \to M$ an equivariant map. Then there exists an invariant $\mathrm{NF}_n^G(f) \in \{0\} \cup \mathbb{N}$ such that

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Dependence of $k \mapsto N_k^G(f)$ on $m$

**Definition 19**

We say that a natural number $r$ *eventually divides* $m$ if $r$ divides a power $m^s$. In other words $r$ eventually divides $m$ if and only if for a prime number $p$

$$p | r \Rightarrow p | m$$

We define $k'$ as the greatest divisor of $k$ dividing eventually $m$.

**Theorem**

Let $G$ be a finite abelian group, $\# G = m$. Then

$$NF_k(f) = NF_{k'}(f)$$

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**Assumption**

Assume that every $k \in \mathbb{N}$ all Reidemeister classes of $\tilde{f}^k$ are essential.

**Theorem**

Let $G = \pi_1 X = \mathbb{Z}_{p_1^{a_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{a_r}}$, where $p_1, \ldots, p_r$ denote different primes, be a cyclic group of order $m = p_1^{a_1} \cdots p_r^{a_r}$. Then for $k$ eventually dividing $m$

$$NF_k^G(f) = \begin{cases} km & \text{if } m \mid k \\ \gcd(m, k) \cdot m & \text{otherwise} \end{cases}$$

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We recall that \( R(\bar{f}^k) = \mathbb{Z}_p \) and \( i_{p^\alpha,p^\beta}[x] = [p^{\alpha-\beta} \cdot x] \). This implies that:

1. all classes in \( OR(\bar{f}^1) = R(\bar{f}^1) = \mathbb{Z}_p \) are irreducible while for \( \alpha \geq 1 \)
2. \( [0] \in R(\bar{f}^{p^\alpha}) \) is reducible and the remaining \( p - 1 \) classes in \( R(\bar{f}^{p^\alpha}) \) are irreducible.

**Proposition**

*Under the above assumptions (and \( \alpha \geq 1 \))*

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NF_{\mathbb{Z}_p}^{p^\alpha}(f) = p + \sum_{\beta}(p^{\beta+2} - p^{\beta+1})
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Let $X$ be a simply-connected finite polyhedron with a free action of a finite group $G$ and $f : X \to X$ be a $G$-equivariant map. Assume that the action of $G$ on $H_\ast(X; \mathbb{Q})$ is trivial. If $\exists$ a prime $p \mid \#G$ and $\exists$ $a \in \mathbb{N}$ such that

$$L(f^{p^a}) \not\equiv 0 \mod (p^{a+1})$$

then $f$ has infinitely many periodic points and

$$\limsup \frac{\#\text{Fix}(f^n)}{n} \geq p.$$


