On projections in the noncommutative 2-torus algebra

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Abstract. We investigate a set of functional equations defining an arbitrary projection in the noncommutative 2-torus algebra $\mathcal{A}_\theta$. The exact solutions of these provide various generalisations of the Power-Rieffel projection. By identifying the corresponding $K_0(\mathcal{A}_\theta)$ classes we get an insight into the general structure of projections in $\mathcal{A}_\theta$.

Key words: noncommutative torus; projections; noncommutative solitons

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1 Introduction

Projections (*i.e.* selfadjoint, idempotent elements) in associative $*$-algebras are main building blocks of the algebraic K-theory. Commutative C^*-algebras, which by Gelfand-Naimark theorem are isomorphic to locally compact Hausdorff spaces, do not contain non-trivial projections. To determine the K_0 group of a unital C^*-algebra $\mathcal{A}$ one thus has to study the equivalence classes of projections in the matrix algebra $M_\infty(\mathcal{A})$. However, when one abandons the assumption of commutativity of the algebra one may encounter various non-trivial projections in the algebra itself which, in some cases, are sufficient to fully determine the group $K_0(\mathcal{A})$.

The K-theory of noncommutative 2-torus algebra A_θ, known also as irrational rotation algebra, has been thoroughly investigated in the 1980's. From the works of Pimsner, Voiculescu and Rieffel (see [14, 15] and references therein) we know that $K_0(A_\theta) \cong \mathbb{Z} \oplus \theta\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$. In the case of noncommutative tori it turns out that projections in the algebra A_θ itself generate the whole group $K_0(A_\theta)$ (see Corollary 7.10 in [17]). The K_0 class of a projection is uniquely determined by its algebraic trace, so any two projections with the same trace must be unitarily equivalent in $M_\infty(A_\theta)$ (see Corollary 2.5 in [15]). On the other hand, it has been already pointed out by Rieffel in [17] that the structure of projections in A_θ is more robust than it would appear from the K-theory level.

The purpose of this paper is to look closer into the structure of projections in A_θ itself. Our main results are summarised in Theorems 2 and 3 in Section 4. The statements are proven by an explicit construction of the relevant projections. The latter may be useful in the applications, where explicit formulas for projections are needed.

The uses of noncommutative tori in physics are multifarious. A most natural one concerns gauge theories developed in terms of finitely generated projective modules, which are noncommutative counterparts of vector bundles [4, 19]. Recently, the projections in the noncommutative torus algebra A_θ gained more interest in the context of string theory [5, 18]. They turned out to be extrema of the tachyonic potential providing solitonic field solutions interpreted in terms of D-branes [1, 10, 11]. Moreover, the projections in A_θ are extensively used in the context of quantum anomalies [6, 13], knot theory [8] or theoretical engineering [12].

The paper is organised as follows: Below we recall some basic definitions to fix notation and make the paper self-contained. In the next section we present a set of functional equations defining an arbitrary projection in A_θ and comment on the adopted method of solving these. In section 3 we investigate some special solutions - the Power-Rieffel type projections. These will serve us firstly to provide an alternative proof of the important Corollary 7.10 form [17]. Secondly, we will use them as a starting point for generalisations to come in section 4. Finally, in section 5, we make conclusions and discuss some open question that arose during the presented analysis.

The algebra of noncommutative 2-torus A_θ is a universal C^*-algebra generated by two unitaries U, V satisfying the following commutation relation

$$VU = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\, UV$$

for some real parameter $\theta \in [0,1)$, which we assume to be irrational. We shall work with $\mathcal{A}_\theta$ - a pre-C^*-algebra [9, 11, 19] of A_θ which is made up of "smooth" elements of the form

$$A_\theta \supset \mathcal{A}_\theta \ni a = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} U^m V^n, \qquad\qquad \{a_{m,n}\} \in \mathcal{S}(\mathbb{Z}^2),$$

where $\mathcal{S}(\mathbb{Z}^2)$ denotes the space of Schwartz sequences on $\mathbb{Z}^2$. It is a standard result [19] that $K_*(\mathcal{A}) \cong K_*(A)$ for any $\mathcal{A}$ - pre-C^*-algebra dense in a C^*-algebra A.

The elements of $\mathcal{A}_\theta$ are conveniently obtained from smooth functions on $\mathbb{T}^2$ with help of the Weyl map [11]. The elements we will be dealing with assume a form (3) so we shall need the restriction of the Weyl map to $S^1 \subset \mathbb{T}^2 = S^1 \times S^1$. It is an injective algebra homomorphism [11] given by

$$\begin{aligned} \rho :\ C^\infty(S^1) \ &\longrightarrow\ \mathcal{A}_\theta, \\ f(x) = \sum_{k\in\mathbb{Z}} f_k\, \mathrm{e}^{2\pi\,\mathrm{i}\,kx} \ &\longmapsto\ f(V) := \rho(f) = \sum_{k\in\mathbb{Z}} f_k\, V^k. \end{aligned}$$

We shall also identify $S^1 \cong \mathbb{R}/\mathbb{Z}$, since it would be more convenient to work with functions on $\mathbb{R}$ periodic with period 1.

The noncommutative 2-torus algebra is equipped with a canonical trace defined by

$$\tau(a) = a_{0,0} = \int_0^1 a_0(x)\,\mathrm{d}x. \tag{1}$$

We shall use it to determine the K_0 class of a projection on the strength of Corollary 2.5 in [15].

Another quantity that may be useful in the study of projections is that of the Chern number [3]

$$c_1(p) = \frac{1}{2\pi i}\tau\big(p\,(\delta_1 p \delta_2 p - \delta_2 p \delta_1 p)\,\big). \tag{2}$$

The operators δ_1, δ_2 are basic derivations of $\mathcal{A}_\theta$, which act on the generators as follows

$$\delta_1 U = 2\pi i U, \qquad \delta_1 V = 0, \qquad \delta_2 U = 0, \qquad \delta_2 V = 2\pi i V.$$

The Chern number is related to the index of a Fredholm operator and thus it is always an integer (see [3, Theorem 11]).

For more details of the noncommutative torus structure the reader may refer to [4, 9, 19].

2 Equations for a general projection in $\mathcal{A}_\theta$

Having recalled the basic features of the noncommutative 2-torus algebra we are ready to investigate the structure of projections in it.

We start with providing a set of functional equations that define a projection in $\mathcal{A}_\theta$. Let us consider the following element of $\mathcal{A}_\theta$

$$p = \sum_{n=-M}^{M} U^n p_n(V), \qquad \text{for some } M \in \mathbb{N}. \tag{3}$$

The conditions for p to be a projection yield a set of functional equations for the functions $p_i \in C^\infty(\mathbb{R}/\mathbb{Z})$

$$p_k(x) = \overline{p_{-k}(x + k\theta)}, \qquad \text{for } k = -M, \dots, M, \tag{4}$$

$$p_k(x) = \sum_{m,a=-M}^{M} p_m(x + a\theta) p_a(x)\, \delta_{m+a,k}, \qquad \text{for } k = -M, \dots, M, \tag{5}$$

$$0 = \sum_{m,a=-M}^{M} p_m(x + a\theta) p_a(x)\, \delta_{m+a,k}, \qquad \text{for } k < -M \text{ and } k > M. \tag{6}$$

Some of the above equations are redundant and the number of independent ones is $3M + 2$. It can be easily seen by noticing that the equations (4-6) with $k < 0$ are equivalent to those with $k > 0$, because the functions p_k with negative indices are actually defined by (4) with $k > 0$. For $M = 0$ formulas (4-6) imply $p_0(x) \equiv 1$ as one may expect. When $M = 1$ one obtains the familiar Power-Rieffel equations [16]. However, for $M \geq 2$ the equations become more and more involved and even the existence of a solution is not obvious. In [6] we found four particular solutions to (4-6) with $M = 2$, which represent different classes of $K_0(\mathcal{A}_\theta)$. In the next sections we present a generalisation of the construction given in [6]. Before we start solving the equations (4-6) let us adopt the following definition.

Definition 2.1. *We say a projection in $\mathcal{A}_\theta$ is of* order M *if it is of the form (3) and $p_M \neq 0$.*

We shall not attempt to provide a general solution to (4-6), but rather present a class of special solutions. Nevertheless, this class turns out to be large enough to accommodate the known projections as well as a number of new ones.

We will consider only real-valued functions although (4) requires only p_0 to be real. Moreover, we have already noted that (4) defines the functions p_k for $k < 0$ and it is convenient to get rid of the functions p_k with negative index k in the equations (5) and (6) before solving them. Our special solutions will be such that each summand on the RHS of (6) is equal to zero independently. The same should hold for summands of (5) with $k > 0$ excluding those with $m = 0$ or $a = 0$ – these are combined to form equations

$$p_k(x)\big(p_0(x) + p_0(x + k\theta) - 1\big) = 0, \tag{7}$$

which we also require to be satisfied independently. The equations (5) with $k < 0$ are redundant and the case $k = 0$ cannot be split into independent equations. After (4) is substituted into (5) for $k = 0$ we obtain

$$p_M^2(x - M\theta) + p_M^2(x) + p_{M-1}^2(x - (M-1)\theta) + p_{M-1}^2(x) + \\ + \dots + p_1^2(x - \theta) + p_1^2(x) + p_0(x)\big(p_0(x) - 1\big) = 0. \tag{8}$$

In the forthcoming sections we provide a systematic method of constructing projections of a given trace (1) and order, that will satisfy the equations (4-6) refined with the above-listed conditions.

For a general projection of the form (3), the Chern number can be written as

$$c_1(p) = \sum_{n=1}^{M} \sum_{k=-M}^{M-n} \int_0^1 dx \Big(n\, \overline{p_{k+n}(x)} \big[p_n(x+k\theta) p_k'(x) - p_n(x) p_k'(x+n\theta) \big] + \\ + \big(\{k,n\} \longleftrightarrow \{-k,-n\} \big) \Big). \tag{9}$$

Under the assumptions listed above the formula (9) simplifies significantly to yield

$$c_1(p) = 6 \int_0^1 dx \sum_{n=1}^{M} n\, p_n(x)^2 p_0'(x).$$

3 Power-Rieffel type projections

We start with recollecting the construction of the Power-Rieffel projection in a slightly more general framework. It will serve us as a starting point for generalisations to come in the next section.

If one sets $p_k = 0$ for all $1 \leq k \leq M-1$ then (4-6) reduce to the Power-Rieffel equations with parameter $M\theta$

$$p_M(x+M\theta) p_M(x) = 0, \tag{10}$$
$$p_M^2(x) + p_M^2(x - M\theta) + p_0(x)(p_0(x) - 1) = 0, \tag{11}$$
$$p_M(x)\big(1 - p_0(x) - p_0(x+M\theta)\big) = 0. \tag{12}$$

A standard solution to (10-12) is known as a Power-Rieffel type projection [7, 11]

$$p_0(x) = \begin{cases} d_M(x), & 0 \leq x \leq \varepsilon_M \\ 1, & \varepsilon_M < x < M\theta \\ 1 - d_M(x - M\theta), & M\theta \leq x \leq M\theta + \varepsilon_M \\ 0, & M\theta + \varepsilon_M < x < 1 \end{cases}, \tag{13}$$

$$p_M(x) = \begin{cases} \sqrt{d_M(x)(1 - d_M(x))}, & 0 \leq x \leq \varepsilon_M \\ 0, & \varepsilon_M < x \leq 1 \end{cases}, \tag{14}$$

where $\theta' = M\theta - \lfloor M\theta \rfloor$ and d_M is a smooth function with $d_M(0) = 0$, $d_M(\varepsilon_M) = 1$. The functions p_0 and p_1 are depicted in figure 1.

Let us now discuss the properties of these projections. First of all, note that due to the periodicity of p_i functions the equations (10-12) are invariant with respect to the transformation $M\theta \to M\theta + z$ for any $z \in \mathbb{Z}$. This means that a Power-Rieffel type projection of order M has the algebraic trace (1) equal to $\theta' = M\theta - \lfloor M\theta \rfloor$. Since θ is irrational we have infinitely many M such that

$$0 < M\theta - n < 1 \qquad \Longleftrightarrow \qquad \frac{n}{M} < \theta < \frac{n+1}{M}.$$

Hence, the following proposition (which is also a consequence of the Corollary 7.10 in [17]) holds.

Proposition 1. *The algebra $\mathcal{A}_\theta$ contains projections representing infinitely many different classes of $K_0(\mathcal{A}_\theta)$.*

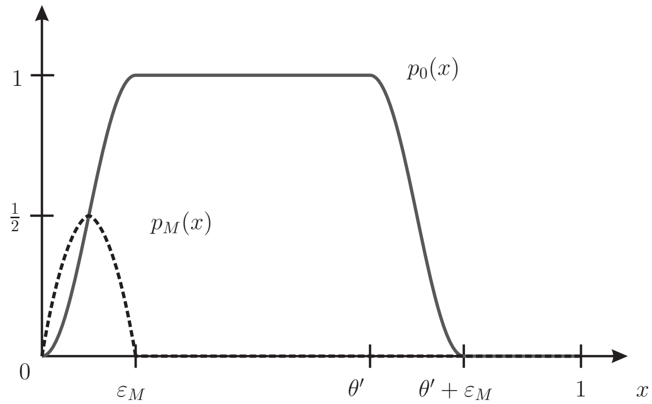

Figure 1. Depiction of functions constituing a Power-Rieffel type projection. We have $0 < \varepsilon_M < \theta'$, $\varepsilon_M + \theta' < 1$.

Another point of view one may adopt for the projection (13-14) is that for any fixed M it is the standard Power-Rieffel projection [16] in the subalgebra of $\mathcal{A}_\theta$ generated by U^M and V. This fact may be used to construct an approximation of $\mathcal{A}_\theta$ in terms of two algebras of matrix valued functions on S^1 [7, 11].

For $p^{[M]}$ - a Power-Rieffel type projection of order M the formula (2) gives

$$c_1(p^{[M]}) = 6M \int_0^{\varepsilon_M} dx\, d_M(x)(1 - d_M(x))d'_M(x) = 6M \left(\tfrac{d_M(x)^2}{2} - \tfrac{d_M(x)^3}{3} \right) \Big|_0^{\varepsilon_M} = M.$$

This is in accordance with the result of [3] stating that if $\tau(p) = |a - b\theta|$ then $c_1(p) = \pm b$.

From the K-theoretic point of view, these projections are sufficient to understand the structure of the equivalence classes of projective modules over $\mathcal{A}_\theta$. On the other hand, the algebra $\mathcal{A}_\theta$ contains other interesting projections, which we shall present in next Section.

4 General projections in $\mathcal{A}_\theta$

Let us now see what kind of projections one can get by letting functions p_k in (3) to be nonzero for some of the indices $k \in \{1, \ldots, M-1\}$. The results are summarised in the following Theorems.

Theorem 2. *A projection of order M may represent the $K_0(\mathcal{A}_\theta)$ class $[n\theta]$, as well as the class $[1 - n\theta]$, for all $n = 1, 2, \ldots, \frac{1}{2}M(M+1)$, provided that $0 < \theta < 1/\max(n, M)$.*

By $[n\theta] \in K_0(\mathcal{A}_\theta)$ we denote the K_0 class represented by a projection $p \in \mathcal{A}_\theta$ with $\tau(p) = n\theta$.

Theorem 3. *The equations (4-6) for a projection of order M admit solutions with $p_k \neq 0$ for every $k \in \{0, \ldots, M\}$ whenever $0 < \theta < 1/M$.*

We shall start with the proof of Theorem 2 by showing how to use the p_k functions to increase or decrease the trace of a Power-Rieffel type projection. Then we present a method of including the remaining p_k functions to the projections constructed in the previous proof without changing its traces. In this way we will prove Theorem 3. Both proofs are constructive so we are able to plot some examples of the p_0 functions of the relevant projections which, as we shall see, determine all of the other functions p_k for $k \neq 0$. A brief discussion of the assumptions limiting the θ parameter may be found in section 5.

Proof of Theorem 2. Let us start with the case of $\tau(p) = n\theta > M\theta$. We shall begin with a Power-Rieffel type projection as defined in the previous section (13, 14). First note that if $M\theta < 1$ then the functions p_0 and p_M of Power-Rieffel type projection vanish for $x \geq M\theta + \varepsilon_M$. If θ is small enough (*i.e.* $(M+k)\theta < 1$) then we can "glue" a Power-Rieffel type projection of trace $k\theta$ to the previous one. Namely, let us keep the definition of p_0 on $[0, M\theta + \varepsilon_M]$ (see (13)) and set

$$p_0(x) = \begin{cases} d_k(x), & M\theta + \varepsilon_M \leq x \leq M\theta + \varepsilon_M + \varepsilon_k \\ 1, & M\theta + \varepsilon_M + \varepsilon_k < x < (M+k)\theta + \varepsilon_M \\ 1 - d_k(x - k\theta), & (M+k)\theta + \varepsilon_M \leq x \leq (M+k)\theta + \varepsilon_M + \varepsilon_k \\ 0, & (M+k)\theta + \varepsilon_M + \varepsilon_k < x < 1 \end{cases},$$
$$p_k(x) = \begin{cases} \sqrt{d_k(x)(1 - d_k(x))}, & M\theta + \varepsilon_M \leq x \leq M\theta + \varepsilon_M + \varepsilon_k \\ 0, & \text{elsewhere} \end{cases},$$

for a smooth function d_k with $d_k(M\theta + \varepsilon_M) = 0$, $d_k(M\theta + \varepsilon_M + \varepsilon_k) = 1$ and a small parameter ε_k. The summands of (5) and (6), which we have assumed to be equal to zero independently, have the form $p_m(x + a\theta)p_a(x)$. This means that all of the non-zero functions p_k for $k \neq 0$ shifted to the interval $x \in [0, \theta]$ must not intersect. The latter can be fulfilled by restricting the ε parameters

$$\varepsilon_M < M\theta, \qquad \varepsilon_k < k\theta, \qquad \varepsilon_M + \varepsilon_k + (M+k)\theta < 1. \tag{15}$$

The imposed restrictions on ε parameters imply that equation (8) reduces to two equations of the form (11). Namely for $x \in [0, \varepsilon_M] \cup [M\theta, M\theta + \varepsilon_M]$ and for $x \in [M\theta + \varepsilon_M, M\theta + \varepsilon_M + \varepsilon_k] \cup [(M+k)\theta + \varepsilon_M, (M+k)\theta + \varepsilon_M + \varepsilon_k]$ we have respectively

$$p_0(x)(1 - p_0(x)) = p_M^2(x) + p_M^2(x - M\theta),$$
$$p_0(x)(1 - p_0(x)) = p_k^2(x) + p_k^2(x - k\theta).$$

These equations are satisfied by construction of p_k and p_M. On the remaining part of the interval $[0,1]$ the equation (8) is trivially satisfied, since both LHS and RHS are equal to 0. By the same argument, equation (7) remains satisfied, as it is satisfied for both Power-Rieffel type projections independently. Thus we have obtained a new projection with a trace $(M+k)\theta$. Examples of p_0 functions defining such projections are depicted in figure 2.

If the parameter θ is small enough (*i.e.* $n\theta < 1$) we can continue the process of "gluing" Power-Rieffel type projections to obtain a projection bearing the trace $n\theta$, with $n \geq M$. If one makes use of all of the functions p_k with $1 \leq k \leq M-1$ to increase the trace, one will end with a projection bearing the trace $(1 + 2 + \ldots + M)\theta = \frac{1}{2}M(M+1)\theta$. The only thing one has to take care of are the conditions satisfied by the parameters ε_k. The restrictions (15) may be easily generalised to the case of non-vanishing p_{k_s} functions with $s \in [1, M-1]$:

$$\varepsilon_{k_j} < k_j\theta, \text{ for } 1 \leq j \leq s, \tag{16}$$
$$\varepsilon_{k_1} + \ldots + \varepsilon_{k_s} + \varepsilon_M + n\theta < 1, \text{ with } n = k_1 + \ldots + k_s + M. \tag{17}$$

Let us note, that the above construction can be obtained (for $n\theta < 1$) by taking a sum of s mutually orthogonal Power-Rieffel type projections $p^{[k_j]}$ of orders k_j. Indeed, one can easily check that the functional equations resulting form the orthogonality conditions

$$(p^{[k_i]})^2 = (p^{[k_i]})^* = p^{[k_i]}, \qquad p^{[k_i]}p^{[k_j]} = p^{[k_j]}p^{[k_i]} = 0, \qquad \text{for } 1 \leq i \neq j \leq s,$$

coincide with the ones derived in subsection 2. A similar construction has been presented in [1].

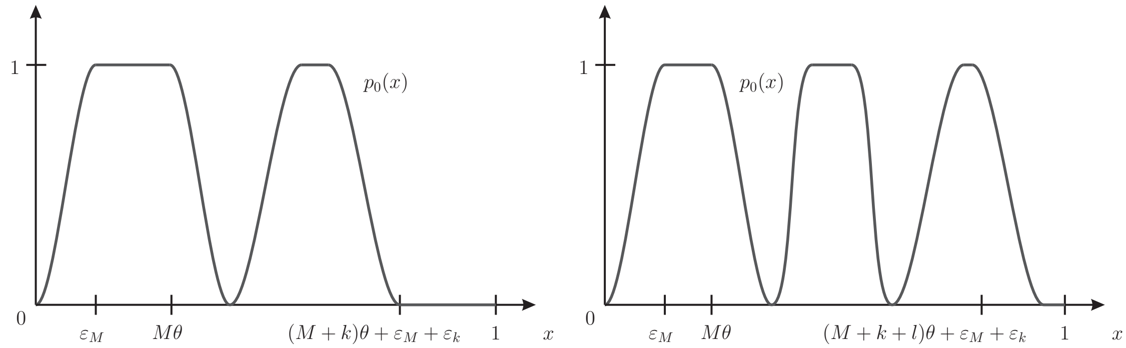

Figure 2. Examples of p_0 functions for projections with traces $(M+k)\theta$ and $(M+k+l)\theta$.

Let us now consider the case of projections of order M and trace $n\theta$ with $1 \leq n < M$. Again, we shall use as a starting point a Power-Rieffel type projection (13, 14), but now we will "cut out" a part of it. Let us set

$$p_0(x) = \begin{cases} d_k(x), & \varepsilon_M \leq x \leq \varepsilon_M + \varepsilon_k \\ 0, & \varepsilon_M + \varepsilon_k < x < k\theta + \varepsilon_M \\ 1 - d_k(x - k\theta), & k\theta + \varepsilon_M \leq x \leq k\theta + \varepsilon_M + \varepsilon_k \\ 1, & k\theta + \varepsilon_M + \varepsilon_k < x < M\theta \end{cases},$$

$$p_k(x) = \begin{cases} \sqrt{d_k(x)(1 - d_k(x))}, & \varepsilon_M \leq x \leq \varepsilon_M + \varepsilon_k \\ 0, & \text{elsewhere} \end{cases},$$

with a smooth function d_k such that $d_k(\varepsilon_M) = 1$, $d_k(\varepsilon_M + \varepsilon_k) = 0$. The conditions $\varepsilon_M < M\theta$, $\varepsilon_k < k\theta$ and $\varepsilon_M + M\theta < 1$ should be satisfied. The situation is now completely analogous to the case of "glued" projections and the same arguments apply. A projection obtained in this way bears the trace $(M-k)\theta$ for $1 \leq k \leq M-1$ (see figure 3).

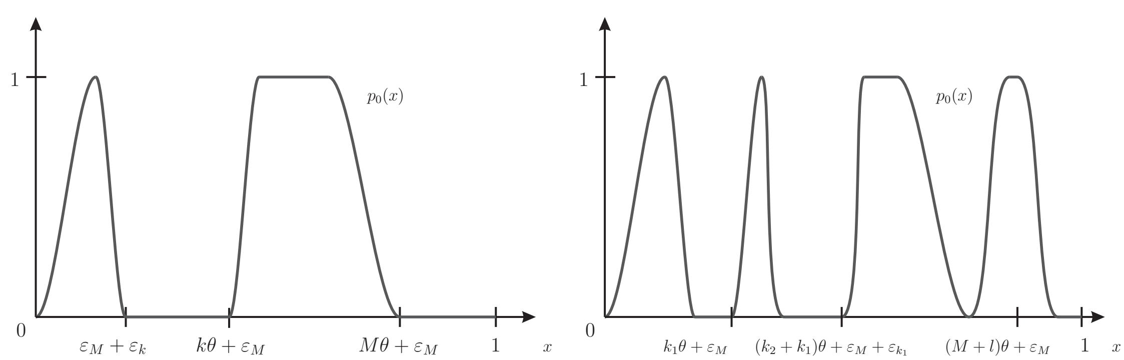

Figure 3. Examples of p_0 functions for projections of trace $(M-k)\theta$ and $(M-k_1-k_2+l)\theta$.

To end the proof of Theorem 2 it remains just to recall that if p is a projection then obviously $1-p$ is so. This means that all of the considerations hold for projections of traces $(1-n\theta)$ - one simply should take $1-p_0$ instead of p_0 and leave p_k for $k \neq 0$ as they are. ∎

The presented proof provides a great variety of possible projections with a given trace, which have, in general, different orders. Let us notice that the two procedures of increasing and decreasing the trace of a projection of a given order can be applied simultaneously and in arbitrary sequence (see figure 3). One only has to choose well the parameters ε_k to have the equations

(16) satisfied. The equations (16) guarantee that the functions d_k do not superpose and the equations (4-6) remain satisfied. This leads us to an enormous number of projections if the order N is big enough. Let us now pass on to the most general projections we were able to construct with the adopted method.

Proof of Theorem 3. In fact one can let all of the p_k function to be non-zero by incorporating to p_0 some "bump functions" d_k. As a starting point, one should take an arbitrary projection defined in section 3 or 4. For sake of simplicity let us now denote by k a free index, *i.e.* we have $p_k = 0$ in our starting point projection. Now, if one sets $p_0(x) = d_k(x)$ for $x \in [\delta_k, \delta_k + \varepsilon_k]$, with $d_k(\delta_k) = d_k(\delta_k + \varepsilon_k) = 1$ or $d_k(\delta_k) = d_k(\delta_k + \varepsilon_k) = 0$ then, to fulfil the equation (7), one has to set $p_0(x) = 1 - d_k(x - k\theta)$ for $x \in [k\theta + \delta_k, k\theta + \delta_k + \varepsilon_k]$. The function p_k should then be defined as previously by $\sqrt{d_k(x)(1 - d_k(x))}$ for $x \in [\delta_k, \delta_k + \varepsilon_k]$ and 0 elsewhere, so that (8) remains fulfilled. The only task to accomplish is to choose well the parameters ε_k and δ_k to avoid the possible intersection of d_k functions. The parameters ε_k should be such that the equations (16) remain satisfied, and $\delta_k = n\theta + \varepsilon_{k_1} + \ldots \varepsilon_{k_s}$ for $n, s \in \mathbb{Z}$ which depend on the concrete projection one has chosen as a starting point. ■

Examples of p_0 functions of the described above projections are shown in figure 4.

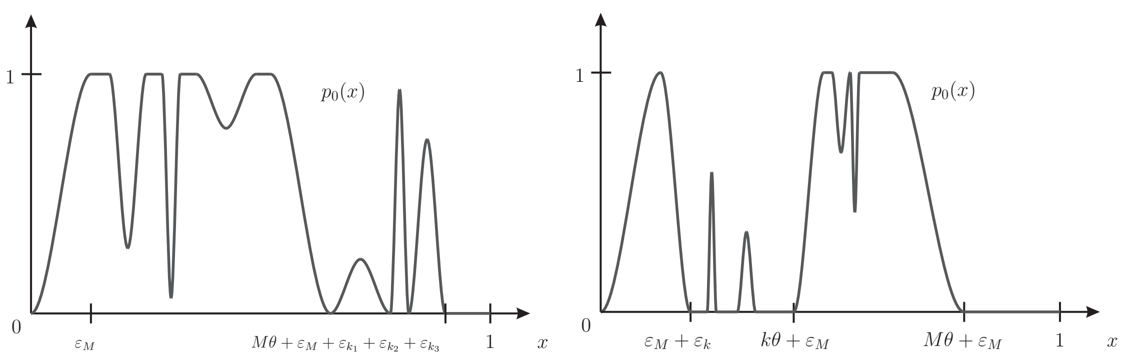

Figure 4. Examples of p_0 functions for projections of traces $M\theta$ and $(M - k)\theta$.

By giving constructive proofs of Theorems 2 and 3 we have exhausted all of the possibilities of constructing projections in $\mathcal{A}_\theta$ with the method described in section 2. To end this Section let us note that the computation of the Chern number of newly constructed projection does not provide any new information. Indeed, it is straightforward either from direct computations of the formula (2), either from an application of the results of [3] that if we have a projection p of trace $n\theta$, then $c_1(p) = n$. In particular, the process of adding "bump" functions described in the proof of Theorem 3 does not change the Chern class of a projection.

Let us now summarise the obtained results and outline the directions of possible further investigations.

5 Conclusion and open questions

In the previous section we have presented many projections, which generalise the standard Power-Rieffel projection. Some represented the same $K_0(\mathcal{A}_\theta)$ class, but had different orders. The others conversely - had the same order, but different traces. A natural question one can ask is what are the relations between the presented projections? The answer is provided by Theorem 8.13 in [17]. It states that if two projections in $\mathcal{A}_\theta$ represent the same $K_0(\mathcal{A}_\theta)$ class (hence have the same trace), then, not only they are unitarily equivalent in $M_\infty(\mathcal{A}_\theta)$, but they are actually in the same path component of the set of projections in $\mathcal{A}_\theta$ itself. This means that there exists a homotopy of projections in $\mathcal{A}_\theta$ for any two of projections in $\mathcal{A}_\theta$ which have the same trace. Indeed, if, for instance, one takes $d_k(t,x) := td_k(x) + (1-t)$ instead of $d_k(x)$ with $d_k(\delta_k) = d_k(\delta_k + \varepsilon_k) = 1$ in the projections constructed at the end of preceding section, then one would obtain a projection for all $t \in [0,1]$.

In consequence, from the topological point of view it is sufficient to consider Power-Rieffel type projection, since they are the generators of the $K_0(\mathcal{A}_\theta)$ group. On the other hand, the richness of the projection structure may show up in applications. In the proof of the Theorem 2 it has already been mentioned that the procedure of "gluing" the Power-Rieffel type projections is in fact equivalent to taking sums of mutually orthogonal projections. However, the "cutting out" described subsequently does not admit an interpretation in terms of subtracting the projections. Indeed, it is straightforward to see that if one expresses a projection p of order M and trace $(M-k)\theta$ as $p = q - r$, where q is a Power-Rieffel type projection of order M, then r would not be a projection. This shows that the newly found projections are not just linear combinations of K_0 generators.

The puzzling thing about the newly found projections is that their existence in $\mathcal{A}_\theta$ seems to depend on the noncommutativity parameter θ as stated in Theorems in section 4. Unfortunately, the solutions presented there cannot be adapted to the case $n\theta > 1$, as it was done for the Power-Rieffel type projections in subsection 3. It is so because the translational symmetry of (10-12), used in the proof of Proposition 1 is absent in general equations (4-6). Note that the discussed symmetry is also broken whenever we introduce the mentioned "bump functions". Whether there is a true difference in the structure of projections in $\mathcal{A}_\theta$ depending on the noncommutativity parameter θ or is it just an artefact of our method of solving (4-6) remains an open question.

To conclude the paper, let us comment on the possible applications of the obtained results to the D-brane scenario in Type II string theories. As mentioned in the Introduction, projections in $\mathcal{A}_\theta$ correspond to solitonic field configurations which are identified with D-branes [1, 10, 11]. On one hand, unitarly equivalent projections yield gauge equivalent field configurations [11, Section 3.1], hence the knowledge of $K_0(\mathcal{A}_\theta)$ alone seems to be sufficient. On the other hand, projections which cannot be written as linear combinations of K_0 generators provide non-perturbative field configurations. Moreover, the homotopy equivalence of projections may be exploit to study the soliton dynamics. An example is provided in [11, Section 6.2], where the Boca projection [2], which is homotopy equivalent to the standard Power-Rieffel projection, is used. The possibility of adding "bump functions" to a projection as described at the end of section 4 indicates the existence of an additional degree of freedom of the D-branes. Finally, let us note that the D-brane point of view suggests that the number of projections in $\mathcal{A}_\theta$ indeed depends on the value of the deformation parameter θ (see [10, Section 4] or [1, Section V]).

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