CUTTING DESCRIPTION OF TRIVIAL 1-COHOMOLOGY.

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We sketch a proof of a following

Theorem 0.0.1. *A connected compact manifold M has trivial first cohomology iff every open and connected set with disconnected boundary cuts M*[1].

Few remarks are in order.

It is an exercise on separation axioms to show that every cutting domain has a disconnected boundary, so one can add that to the theorem without harm.

We have published a result for $\mathbb{R}^n$: every domain with disconnected boundary cuts the ambient manifold[2]. The proof is valid for simply connected metric spaces, and the theorem is true for arbitrary simply connected spaces that share a simple feature: connected open sets are path-connected.

The published theorem has some applications to complex analysis. Suprisingly, as simple as it is, it was nowhere to be found in literature.

The present theorem should also be valid for more general spaces - but probably only as general as arbitrary CW-coplexes.

PROOF that trivial first cohomology implies cutting.

Suppose we have a manifold M, open, connected subset U which has disconnected boundary $\partial U = A \sqcup B$. The only nontrivial situation is when $M \setminus U$ has nonempty, connected interior V and we have

$$M = U \cup \widetilde{A} \cup V \cup \widetilde{B}$$

with $\widetilde{A}$ and $\widetilde{B}$ - disjoint compact neighborhoods of appropriate sets. Now define a map to a circle - two intervals (L-eft and R-ight) glued with N-orth and S-outh pole:

$$f(\widetilde{A}) = N$$

$$f(\widetilde{B}) = S$$

$$f(x) = \frac{dist_C(x, B)}{dist_C(x, \widetilde{B}) + dist_C(x, \widetilde{A})} \in [0,1]_L \;\; x \in U$$

[1] I.e., $M \setminus U$ is not connected.

[2] A. Czarnecki, W. Lubawski, M. Kulczycki, "On connectivity of boundary and complement for domains", Ann. Polon. Math. 103, 2011

$$f(x) = \frac{dist_C(x,A)}{dist_C(x,\widetilde{B}) + dist_C(x,\widetilde{A})} \in [0,1]_R \;\; x \in V$$

with $dist$ counted in path metric d_C: $d_C(x,y) := inf\{$lenght of paths joining x and $y\}$. This is continuous in path metric topology, which is fortunately equivalent to the standard one. Now take a loop γ joining a point in A with a point in B through V and going back through U. Now a map $f_* \circ \gamma_*$ is, by construction, nonzero as an endomorphism of $\pi_1(\mathbb{S}^1)$. But f_* must be zero, moreover - nullhomotopic, a contradiction. $\square$

The fact that f must be nullhomotopic follows from:

$$[M, \mathbb{S}^1] = [M, K(\mathbb{Z}, 1)] = H^1(M, \mathbb{Z}) = 0$$

where the isomorphisms are respectively: $\mathbb{S}^1$ is Eilenberg-MacLane space and homotopy characterisation of CW-complex' cohomology. This allows us to start

PROOF that cutting implies trvival cohomology.

Suppose that M has a nontrivial first cohomology class, which is classified by a homotopy class of maps from M to $\mathbb{S}^1$. Pick a smooth map in this class, moreover - as Morse theory is local in nature - pick a Morse function. Observe that we can choose this function having critical points neither of index 0 nor maximal. Now choose a regular value v and cut the manifold along the preimage of v, obtaining essentially a honest Morse function $f : M \to [0,1]$. Keeping in mind that (reverse) gradient flow that hits the top level set comes around the bottom, we proceed as follows. Pick a interval $[a,b]$, consisting only of regular values. The preimage $f^{-1}[a,b]$ decomposes as a finite disjoint union of connected cylinders over components of $f^{-1}(a)$. Now pick any point on the manifold and travel along the (reverse) gradient flow (remembering that the top and bottom sets are identified). As the function has no local extrema, most trajectories hit every level set infinitely many times. As there is only finite number of components, a trajectory must eventually come twice to some component of $f^{-1}[a,b]$. This one is precisely a connected open set with disconnected boundary that does not cut the ambient manifold. $\square$

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