# TRIVIAL ALEXANDER-SPANIER 1-COHOMOLOGY AND BOUNDARY OF A DOMAIN.

ANDRZEJ CZARNECKI

In [1] we have shown that a compact connected manifold $M$ has trivial $H^1(M,\mathbb{Z})$ if and only if every connected open subset $U$ leaves $M \setminus U$ disconnected if it has disconnected boundary. We use here singular cohomology.

We will examine the "only if" implication in the case of Alexander-Spanier cohomology. The reader is kindly asked not to assume that this is merely a repetition, we investigate different property with different methods.

Consider a triple $(M \setminus U, \overline{U}, \partial U)$, where $M$ is an arbitrary manifold (as in: not necessarily compact, but still connected) with $H^1_{AS}(M,\mathbb{Z})$ trivial. Each and every algebraic topologist will now write down the Mayer-Vietoris sequence – it's zeroth part encodes the conectedness of the spaces in question:

$$0 \longrightarrow H^0_{AS}(M,\mathbb{Z}) \longrightarrow H^0_{AS}(\overline{U},\mathbb{Z}) \oplus H^0_{AS}(M\setminus U,\mathbb{Z}) \longrightarrow$$
$$\longrightarrow H^0_{AS}(\partial U,\mathbb{Z}) \longrightarrow 0 = H^1_{AS}(M,\mathbb{Z})$$

Sadly, this is not a Mayer-Vietoris sequence: it is not exact, as the triple in question might not be an excisive triad. However, Alexander-Spanier cohomology is rigid: the poset of open neighbourhoods of the sets in our triple gives a direct system of Alexander-Spanier cohomology groups and this system has a limit in Alexander-Spanier cohomology of the original sets[1]. We only need to pic a cofinite subposet to obtain the limit, so for the sake of brevity, assume $\partial U$ compact and choose the $\epsilon$-envelopes: $A^\epsilon = \bigcup_{a\in A} B(a,\epsilon)$, the ball taken in a metric compatible with te topology and chosen somewhere along the way. We can now write an actual Mayer-Vietoris sequence:

$$0 \longrightarrow H^0_{AS}(M,\mathbb{Z}) \longrightarrow H^0_{AS}(\overline{U}^\epsilon,\mathbb{Z}) \oplus H^0_{AS}((M\setminus U)^\epsilon,\mathbb{Z}) \longrightarrow$$
$$\longrightarrow H^0_{AS}(\partial U^\epsilon,\mathbb{Z}) \longrightarrow 0 = H^1_{AS}(M,\mathbb{Z})$$

As we go with $\epsilon$ to 0, the groups converge to the groups in the first sequence, and although the exactness may break for the limit sequence, we are happy just with seeing that an additional $\mathbb{Z}$ in the middle term emerges (we rewrite the sequence, but – again – do not claim it to be exact) if and only if it does also in the $\partial U$ part:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \longrightarrow H^0_{AS}(\partial U,\mathbb{Z}) \longrightarrow 0$$

[1]Of course, there is some assumptions on spaces to be satisfied if we want to hope for rigidity. On sufficient assumption is that the ambient space should be paracompact and the subset in question closed, which happens to be the case here.

Then it must show somwhere earlier in the direct system, proving some envelope of the boundary disconnected, hence also the boundary itself. A similar argument can be given for noncompact boundary.

Observing what we have actually did, we can state our theorem in full generality:

**Theorem 0.0.1.** *Given a locally connected metric space $X$ with $H^1_{AS}(X) = 0$, an open subset $U$ makes $M \setminus U$ disconnected if and only if it has a disconnected boundary.*

We conclude with a philosophical remark that presents itself upon investigating the subject: trivial 1-cohomology allows the passage from local (connected basis) dimension-theory data (cutting by appropriate subspaces) to global data (open domains of arbitrary size).

## References

[1] A. Czarnecki, *Cutting description of trivial 1-cohomology*, arXiv:1206.1811v1 [math.AT]

Department of Mathematics and Computer Sciences, Jagiellonian University
*E-mail address*: andrzejczarnecki01@gmail.com