A NEW HARTOGS TYPE EXTENSION THEOREM FOR THE CROSS-LIKE OBJECTS

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Abstract. The main result of the paper is a new Hartogs type extension theorem for the $\mathscr{A}$-crosses, which generalizes the extension theorem for the (N,k)-crosses (see [6]), as well as the classical Cross Theorem (see [1]).

1. Introduction

The celebrated Hartogs theorem ([3]) states that every separately holomorphic function in several complex variables is necessarily holomorphic as a function of all variables. This over one hundred years old deep result has been generalized in many directions. One of them leads to the theory of the extension of separately holomorphic functions defined on the so-called crosses ([1]). The newest results in this area concern the (N,k)-crosses ([6]) and the generalized (N,k)-crosses ([9],[13]). We introduce the $\mathscr{A}$-crosses, which generalize the (N,k)-crosses in a different way than the generalized (N,k)-crosses: for the first time the cross-like objects admit the different sizes of the branches - for the details see Definition 3.3.

When the cross-like objects are concerned, two basic questions appear in a natural way. The first question is, whether all separately holomorphic functions defined on such objects extend to some open neighborhood of them. If the answer is positive, we can consider the problem of finding a nice description of the envelope of holomorphy of the cross-like object (for instance, in terms of the relative extremal function; see for example the case of (N,k)-crosses, [6]). Our Main Theorem (Theorem 3.8) says that the envelope of holomorphy of the $\mathscr{A}$-cross is exactly the envelope of holomorphy of some

2010 *Mathematics Subject Classification.* Primary 32D15; Secondary 32D10.

Key words and phrases. Hartogs theorem, separately holomorphic functions, $\mathscr{A}$-crosses.

Project operated within the Foundation for Polish Science IPP Programme "Geometry and Topology in Physical Models" co-financed by the EU European Regional Development Fund, Operational Program Innovative Economy 2007-2013.

corresponding classical 2-fold cross containing it, if only we exclude some "bad" case. Concerning the second question, in Section 4 we present how various such descriptions can be.

The paper was partially written during the author's stay at the Carl von Ossietzky Universität Oldenburg. The author would like to express his gratitude to Professor Peter Pflug for his constant help and inspiring discussions. The author also owes thanks to Professor Marek Jarnicki for his commitment and many valuable remarks.

2. Prerequisites and notations

The natural objects treated in this article are Riemann regions. The interested reader is asked to consult [4] for a wide exposition of the theory of Riemann regions. In the present paper $\mathcal{PLP}(X)$ stands for the family of all pluripolar subsets of an arbitrary Riemann region X and $\mathcal{O}(X)$ is the space of all holomorphic functions on X; furthermore, by $\mathcal{PSH}(X)$ we will denote the family of all plurisubharmonic functions on X.

Definition 2.1 ([7], Chapter 3)**.** Let X be a Riemann region over $\mathbb{C}^n$ and let $A \subset X$. The *relative extremal function of A with respect to X* is the upper semicontinuous regularization $h^\star_{A,X}$ of the function

$$h_{A,X} := \sup\{u : u \in \mathcal{PSH}(X), u \leq 1, u|_A \leq 0\}.$$

Definition 2.2 ([7], Chapter 3)**.** We say that a set $A \subset X$ is *pluriregular at a point $a \in \overline{A}$* if $h^\star_{A,U}(a) := h^\star_{A\cap U,U}(a) = 0$ for any open neighborhood U of a. Define

$$A^\star = A^{\star,X} := \{a \in \overline{A} : A \text{ is pluriregular at } a\}.$$

We say that A is *locally pluriregular* if $A \neq \varnothing$ and A is pluriregular at each of its points, i.e. $\varnothing \neq A \subset A^\star$.

It is a simple observation that if A is locally pluriregular, then $h^\star_{A,X} \equiv h_{A,X}$. In that case we shall omit the star when using the relative extremal function. For a good background on the topic of the relative extremal function we refer the reader to [7].

3. $\mathscr{A}$-crosses and the main result

Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $\varnothing \neq A_j \subset D_j$ for $j = 1, \dots, N$, $N \geq 2$.

For $k \in \{1, \dots, N\}$ let $I(N,k) := \{\alpha = (\alpha_1, \dots, \alpha_N) \in \{0,1\}^N : |\alpha| = k\}$, where $|\alpha| := \alpha_1 + \ldots + \alpha_N$.

Put

$$\mathcal{X}_{\alpha,j} := \begin{cases} D_j, & \text{if } \alpha_j = 1 \\ A_j, & \text{if } \alpha_j = 0 \end{cases}, \quad \mathcal{X}_\alpha := \prod_{j=1}^{N} \mathcal{X}_{\alpha,j}.$$

For an $\alpha \in I(N,k)$ such that $\alpha_{r_1} = \ldots = \alpha_{r_k} = 1, \alpha_{i_1} = \ldots = \alpha_{i_{N-k}} = 0$, where $r_1 < \ldots < r_k$ and $i_1 < \ldots < i_{N-k}$, put

$$D_\alpha := \prod_{s=1}^{k} D_{r_s}, \quad A_\alpha := \prod_{s=1}^{N-k} A_{i_s}.$$

For every $\alpha \in I(N,k)$ and every $a = (a_{i_1}, \dots, a_{i_{N-k}}) \in A_\alpha$ define the mapping

$$\boldsymbol{i}_{a,\alpha} = (\boldsymbol{i}_{a,\alpha,1}, \dots, \boldsymbol{i}_{a,\alpha,N}) : D_\alpha \to \mathcal{X}_\alpha,$$

$$\boldsymbol{i}_{a,\alpha,j}(z) := \begin{cases} z_j, & \text{if } \alpha_j = 1 \\ a_j, & \text{if } \alpha_j = 0 \end{cases}, \quad j = 1, \dots, N, \quad z = (z_{r_1}, \dots, z_{r_k}) \in D_\alpha$$

(if $\alpha_j = 0$, then $j \in \{i_1, \dots, i_{N-k}\}$ and if $\alpha_j = 1$, then $j \in \{r_1, \dots, r_k\}$).

In [6] Jarnicki and Pflug introduced the so-called (N,k)-crosses.

Definition 3.1. An (N,k)-*cross* is defined as

$$\mathbf{X}_{N,k} = \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N) := \bigcup_{\alpha \in I(N,k)} \mathcal{X}_\alpha.$$

The *envelope of an* (N,k)-*cross* is defined as

$$\hat{\mathbf{X}}_{N,k} = \hat{\mathbb{X}}_{N,k}((A_j, D_j)_{j=1}^N) := \\ \{(z_1, \dots, z_N) \in D_1 \times \ldots \times D_N : \sum_{j=1}^{N} h^\star_{A_j^\star, D_j}(z_j) < k\}.$$

In particular, if all A_j's are locally pluriregular, then $\mathbf{X}_{N,k} \subset \hat{\mathbf{X}}_{N,k}$.

Observe that for $k = 1$ the above definition leads to the classical N-fold crosses ([5]). In the case where $N = 2$ sometimes it will be more convenient to use the notation $\mathbb{X}(A_1, A_2; D_1, D_2) := \mathbb{X}_{2,1}((A_j, D_j)_{j=1,2})$. In the sequel we shall intensively use the following

Theorem 3.2 ([6]). *For every function f, separately holomorphic on $\mathbf{X}_{N,k}$ (cf. Definition 3.5), there exists an $\hat{f} \in \mathcal{O}(\hat{\mathbf{X}}_{N,k})$ such that $\hat{f} = f$ on $\mathbf{X}_{N,k}$ and $\hat{f}(\hat{\mathbf{X}}_{N,k}) \subset f(\mathbf{X}_{N,k})$.*

After the (N,k)-crosses the question arises, in which direction should we go to consider more general objects. In [9] the so-called generalized (N,k)-crosses are concerned. Here we shall introduce the $\mathscr{A}$-crosses.

Definition 3.3. Fix a natural number $N \geq 2$. For any $\varnothing \neq S \subset \{1,\dots,N\}$ we choose some system of multiindices $\alpha(s)^1,\dots,\alpha(s)^{l_s} \in I(N,s), s \in S$. For $s \in \{1,\dots,N\} \setminus S$ put $l_s = 0$. In the set $\{\alpha(s)^r : s \in S, r = 1,\dots,l_s\}$ consider the lexicographical order and denumarate its elements with respect to this order as $\alpha^1,\dots,\alpha^{l_1+\dots+l_N}$. Build the matrix $\mathscr{A} := (\alpha^i_j)_{j=1,\dots,N, i=1,\dots,l_1+\dots+l_N}$ and define the *$\mathscr{A}$-cross*

$$\mathbf{Q}(\mathscr{A}) = \mathbf{Q}(\alpha^i_j) = \mathbb{Q}(\alpha^i_j)((A_j,D_j)_{j=1}^N) := \bigcup_{s\in S}\bigcup_{r=1}^{l_s} \mathcal{X}_{\alpha(s)^r}.$$

We say that the sets $\mathcal{X}_{\alpha(s)^r}$ are *branches* of $\mathbf{Q}(\mathscr{A})$.

The $\mathscr{A}$-cross is said to be *reduced*, if there is no nontrivial chain (with respect to the lexicographical order) in the set $\{\alpha(s)^r : r = 1,\dots,l_s, s \in S\}$ (i.e. there is no situation where some branch $\mathcal{X}_{\alpha(s_1)^{r_1}}$ is essentially contained in some another branch $\mathcal{X}_{\alpha(s_2)^{r_2}}$).

From now on, without loss of generality, we shall consider only the reduced $\mathscr{A}$-crosses.

Example 3.4. For $N = 2$ we have the following $\mathscr{A}$-crosses:

$$\mathbf{Q}\begin{pmatrix}1 & 0\end{pmatrix} = D_1 \times A_2,$$
$$\mathbf{Q}\begin{pmatrix}0 & 1\end{pmatrix} = A_1 \times D_2,$$
$$\mathbf{Q}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = (A_1 \times D_2) \cup (D_1 \times A_2) = \mathbb{X}_{2,1}((A_j,D_j)_{j=1,2}),$$
$$\mathbf{Q}\begin{pmatrix}1 & 1\end{pmatrix} = D_1 \times D_2 = \mathbb{X}_{2,2}((A_j,D_j)_{j=1,2}).$$

For $N = 3$ we have the following $\mathscr{A}$-crosses (up to permutations of variables):

$$\mathbf{Q}\begin{pmatrix}0 & 1 & 1\end{pmatrix} = A_1 \times D_2 \times D_3,$$
$$\mathbf{Q}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 0\end{pmatrix} = (A_1 \times D_2 \times D_3) \cup (D_1 \times A_2 \times A_3) = \mathbb{X}(A_1, A_2 \times A_3; D_1, D_2 \times D_3),$$
$$\mathbf{Q}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\end{pmatrix} = (A_1 \times D_2 \times D_3) \cup (D_1 \times A_2 \times D_3) = \mathbb{X}_{2,1}((A_j,D_j)_{j=1,2}) \times D_3 = \mathbb{X}(A_1, A_2 \times D_3; D_1, D_2 \times D_3),$$

$$\mathbf{Q}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \mathbb{X}_{3,2}((A_j, D_j)_{j=1,2,3}),$$

$$\mathbf{Q}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = (A_1 \times A_2 \times D_3),$$

$$\mathbf{Q}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (A_1 \times A_2 \times D_3) \cup (D_1 \times A_2 \times A_3),$$

$$\mathbf{Q}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbb{X}_{3,1}((A_j, D_j)_{j=1,2,3}),$$

$$\mathbf{Q}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = (D_1 \times D_2 \times D_3) = \mathbb{X}_{3,3}((A_j, D_j)_{j=1,2,3}).$$

Observe that in this new notation the set of (N, k)-crosses equals the set of those $\mathscr{A}$-crosses, for which the rows of the defining matrix $\mathscr{A}$ are exactly the elements of $I(N, k)$ ordered lexicographically. The notion of separate holomorphicity is completely in the spirit of (N, k)-crosses (see [6]).

Definition 3.5. We say that a function $f : \mathbf{Q}(\alpha^i_j) \to \mathbb{C}$ is *separately holomorphic on* $\mathbf{Q}(\alpha^i_j)$ if for every $s \in \{1, \ldots, N\}$ with nonzero l_s, every $j \in \{1, \ldots, l_s\}$, and for every $a \in A_{\alpha(s)^j}$ the function

$$D_{\alpha(s)^j} \ni z \mapsto f(\boldsymbol{i}_{a,\alpha(s)^j}(z))$$

is holomorphic.

As always we ask whether any separately holomorphic function on an $\mathscr{A}$-cross $\mathbf{Q}$ can be extended holomorphically to some open neighborhood of $\mathbf{Q}$. The second question is, whether we can effectively describe the envelope of holomorphy of an $\mathscr{A}$-cross. Note that usually $\mathscr{A}$-crosses are not open. However, if any separately holomorphic function on an $\mathscr{A}$-cross $\mathbf{Q}$ extends holomorphically to some open neighborhood of $\mathbf{Q}$, we may speak of envelope of holomorphy of that naighborhood. Therefore, the problem of finding the *envelope of holomorphy of a given $\mathscr{A}$-cross* should not lead to confusions.

As the following example shows, we need to avoid some "pathological" situation.

Example 3.6. Take

$$\mathbf{Q} := \mathbf{Q}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (A_1 \times A_2 \times D_3) \cup (D_1 \times A_2 \times A_3),$$

Consider on $\mathbf{Q}$ the function $f(z_1, z_2, z_3) := g(z_2)h(z_1, z_3)$, where h is some separately holomorphic function on $\mathbf{X} := \mathbb{X}(A_1, A_3; D_1, D_3)$ (possibly constant; it extends holomorphically to $\hat{\mathbf{X}}$) and g is some "wild" function on

A_2. Then f is separately holomorphic on $\mathbf{Q}$. It may be, however, very far away from being holomorphically extendible (or just holomorphic, in the case where $A_j = D_j, j = 1, 2, 3$).

Example 3.7. Consider

$$\mathbf{Q} := \mathbf{Q}(1 \quad \cdots \quad 1) = \mathbb{X}_{N,N}((A_j, D_j)_{j=1}^N) = D_1 \times \ldots \times D_N.$$

Then every separately holomorphic function on $\mathbf{Q}$ is authomatically holomorphic on $\mathbf{Q}$. If in addition all D_j's are domains of holomorphy, then so is $\mathbf{Q}$.

The following theorem - our Main Theorem - shows that if we exclude the situation where there exists some $k \in \{1, \ldots, N\}$ such that for any $s \in \{1, \ldots, N\}$ with nonzero l_s and for any $\alpha(s)^r, r = 1, \ldots, l_s$ we have $\alpha(s)^r_k = 0$ (in other words, we assume the inclusion $\mathbb{X}_{N,1}((A_j, D_j)_{j=1}^N) \subset \mathbb{Q}(\alpha^i_j)((A_j, D_j)_{j=1}^N))$, then we may describe the envelope of holomorphy of an $\mathscr{A}$-cross as the envelope of holomorphy of some corresponding 2-fold cross.

Theorem 3.8. *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$ and let $A_j \subset D_j$ be locally pluriregular, $j = 1, \ldots, N$. Put $\mathbf{Q} := \mathbb{Q}(\alpha^i_j)((A_j, D_j)_{j=1}^N)$. Assume that $\mathbb{X}_{N,1}((A_j, D_j)_{j=1}^N) \subset \mathbf{Q}$. Then there exist an $m_0 \in \{1, \ldots, N\}$, a domain of holomorphy $G \subset D_1 \times \ldots \times D_{m_0-1} \times D_{m_0+1} \times \ldots \times D_N$, a locally pluriregular subset $B \subset G$, and a 2-fold classical cross $\mathbf{X} = \mathbb{X}(A_{m_0}, B; D_{m_0}, G)$ containing $\tau^{-1}(\mathbf{Q}_{N;l_1,\ldots,l_N}(\alpha^i_j))$, where τ is a mapping which sends a point $(z_{m_0}, z_1, \ldots, z_{m_0-1}, z_{m_0+1}, \ldots, z_N) \in D_{m_0} \times D_1 \times \ldots \times D_{m_0-1} \times D_{m_0+1} \times \ldots \times D_N$ to the point $(z_1, \ldots, z_N) \in D_1 \times \ldots \times D_N$, such that for every function $f \in \mathcal{F} := \mathcal{O}_s(\mathbf{Q})$ there exists a unique function $\hat{f} \in \mathcal{O}(\hat{\mathbf{X}})$ with $\hat{f} = f \circ \tau$ on $\tau^{-1}(\mathbf{Q})$, i.e. $\hat{\mathbf{Q}} := \tau(\hat{\mathbf{X}})$ is the envelope of holomorphy of $\mathbf{Q}$.*

Proof. The proof is by induction on N. For $N = 2$ the conclusion is obvious (see Example 3.4). Suppose now that the conclusion holds true for $N - 1$ with some $N \geq 3$ and consider $\mathbf{Q}(\alpha^i_j)$.

Let $\mathfrak{K}$ be the set of those $k \in \{1, \ldots, N\}$ such that for any $s \in \{1, \ldots, N\}$ with nonzero l_s and for any $\alpha(s)^r, r = 1, \ldots, l_s$ we have $\alpha(s)^r_k = 1$. Two cases have to be considered.

Case 1. There is some $k_0 \in \mathfrak{K}$.

Observe that if $\mathfrak{K} = \{1, \ldots, N\}$, then $\mathbf{Q} = D_1 \times \ldots \times D_N$ and we take

$m_0 = 1, G = B = D_2 \times \ldots \times D_N, \mathbf{X} = \hat{\mathbf{X}} = D_1 \times \ldots \times D_N$.

Therefore, we may assume that there is some $m_0 \in \{1, \ldots, N\} \setminus \mathfrak{K}$.

To simplify the notation we assume that $k_0 = N \in \mathfrak{K}$ (the proof in the other cases goes along the same lines). Then we see that $\mathbf{Q}(\alpha^i_j) = \mathbf{Q}' \times D_N$, where $\mathbf{Q}'$ is some $\mathscr{A}$-cross, with the defining matrix of dimension $(l'_1 + \ldots + l'_{(N-1)}) \times (N-1)$. By the inductive assumption and Terada's theorem ([12]), for every function from $\mathcal{F}$, the function $f \circ \tau$ extends holomorphically to $\hat{\mathbb{X}}(A_{m_0}, B'; D_{m_0}, G') \times D_N$ with some domain of holomorphy $G' \subset D_1 \times \ldots \times D_{m_0-1} \times D_{m_0+1} \times \ldots \times D_{N-1}$ and locally pluriregular set $B' \subset G'$. It is left to observe that $\hat{\mathbb{X}}(A_{m_0}, B'; D_{m_0}, G') \times D_N = \hat{\mathbb{X}}(A_{m_0}, B' \times D_N; D_{m_0}, G' \times D_N)$.

Case 2. The set $\mathfrak{K}$ is empty.

We use the following notation: for any s with nonzero l_s and any $r \in \{1, \ldots, l_s\}$ write the multiindex $\alpha(s)^r$ as $(\alpha(s)^r_1, \alpha(s)^r_0)$, where $\alpha(s)^r_0$ is a suitable multiindex from $\{0,1\}^{N-1}$.

Fix a point $a_1 \in A_1$ and consider the family of functions $\mathcal{F}' := \{f(a_1, \cdot) : f \in \mathcal{F}\}$. Observe that the functions from $\mathcal{F}'$ are defined on an $\mathscr{A}$-cross $\mathbf{Q}'(\alpha'^i_j)$, where the defining matrix $(\alpha'^i_j)_{j=1,\ldots,N-1, i=1,\ldots,l'_1+\ldots+l'_{N-1}}$ comes from the set of all $\alpha(s)^r_0$'s after ordering its elements with respect to the lexicographical order. The $\mathscr{A}$-cross $\mathbf{Q}'(\alpha'^i_j)$, however, may not be reduced. Nevertheless, the cancelling all the branches of it, which are contained in another ones, does not change anything here, so we may assume without loss of generality that $\mathbf{Q}'(\alpha'^i_j)$ is reduced. By the inductive assumption (observe that all assumptions of the theorem are now satisfied), for any function $g \in \mathcal{F}'$, the function $g \circ \sigma$ extends holomorphically to $\hat{\mathbf{X}}' = \hat{\mathbb{X}}'(A_{m'_0}, B'; D_{m'_0}, G')$ with some domain of holomorphy $G' \subset D_2 \times \ldots \times D_{m'_0-1} \times D_{m'_0+1} \times \ldots \times D_N$ and locally pluriregular set $B' \subset G'$, where $\sigma : D_{m'_0} \times D_2 \times \ldots \times D_{m'_0-1} \times D_{m'_0+1} \times \ldots \times D_N \to D_2 \times \ldots \times D_N$ sends a point $(z_{m'_0}, z_2, \ldots, z_{m'_0-1}, z_{m'_0+1}, \ldots, z_N)$ to the point $(z_2, \ldots, z_N)$.

Furthermore, there exists a minimal number $n_0 \in \{1, \ldots, l_1 + \ldots + l_N\}$ such that for every $n' \geq n_0$ there is $\alpha^{n'}_1 = 1$ while for $n'' < n_0$ we have $\alpha^{''}_1 = 0$. Fix a point $z' \in \mathbf{Q}'' := \mathbf{Q}''(\alpha''^i_j)$, where the matrix $(\alpha''^i_j)_{j=1,\ldots,N-1, i=1,\ldots,l''_1+\ldots+l''_{N-1}}$ comes from the set $\{\alpha(s)^{n'}_0 : n' \geq n, l_s \neq 0\}$ after ordering its elements with respect to the lexicographical order. Also, similarly as above, we may assume that $\mathbf{Q}''$ is reduced. Consider the family of functions $\mathcal{F}'' := \{f(\cdot, z') : f \in \mathcal{F}\}$. Then every function from $\mathcal{F}''$ is holomorphic on D_1.

Observe that $\mathbf{Q}'' \subset \sigma(\hat{\mathbf{X}}')$, since by the construction, for any row α''^i of the matrix (α''^i_j) there exists some row α'^k of the matrix (α'^i_j) which is subsequent to α''^i with respect to the lexicographical order (i.e. any branch of $\mathbf{Q}''$ is contained in some branch of $\mathbf{Q}'(\alpha'^i_j)$). Now the conclusion holds true because of Theorem 3.2 applied to the 2-fold cross $(A_1 \times \sigma(\hat{\mathbf{X}}')) \cup (D_1 \times \mathbf{Q}'')$ and $m_0 = 1, G = \sigma(\hat{\mathbf{X}}'), B = \mathbf{Q}'', \tau = id$ are good for our purpose. $\square$

Remark 3.9. Observe that in the above theorem (and its proof) the set of m_0's that we can distinct can have more than one element. However, since $\hat{\mathbf{X}}$ is the envelope of holomorphy of $\mathbf{Q}$, the result must be the same, no matter which m_0 we distinct.

To see this, observe that if $\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2$ are two envelopes of $\mathbf{Q}$ constructed for two different m_0's, then for any function $f \in \mathcal{O}(\hat{\mathbf{X}}_1)$ there exists an $\tilde{f} \in \mathcal{O}(\hat{\mathbf{X}}_2)$ with $\tilde{f} = f$ on the connected componnent of $\hat{\mathbf{X}}_1 \cap \hat{\mathbf{X}}_1$ containing $\mathbf{Q}$ (notice that $\mathbf{Q}$ is connected - this follows from the proof of Theorem 3.8 and main result from [9]).

Corollary 3.10. *Let $s, N_1, \dots, N_s \in \mathbb{N}, s, N_1, \dots, N_s \geq 2$, let D_{d_j} be a Riemann domain of holomorphy over $\mathbb{C}^{n_{d_j}}$ and $A_{d_j} \subset D_{d_j}$ be locally pluriregular, $j = 1, \dots, N_d, d = 1, \dots, s$. Let $\mathbf{Q} = \mathbf{Q}_1 \times \dots \times \mathbf{Q}_s$, where $\mathbf{Q}_d$ is an $\mathscr{A}$-cross spread over $D_{d_1} \times \dots \times D_{d_{N_d}}, d = 1, \dots, s$. Assume that $\mathbb{X}_{N_1+\dots+N_s,1}((A_j, D_j)_{j=1}^{N_1+\dots+N_s}) \subset \mathbf{Q}$. Then the envelope of holomorphy $\hat{\mathbf{Q}}$ of $\mathbf{Q}$ equals $\hat{\mathbb{X}}_{s,1}((\mathbf{Q_j}, \hat{\mathbf{Q}}_j)_{j=1}^s)$.*

Proof. Without loss of generality we may assume that $\mathbf{Q}_j$ is reduced, $j = 1, \dots, s$.

Fix an $f \in \mathcal{F} := \mathcal{O}_s(\mathbf{Q})$. Observe that then for any $j \in \{1, \dots, s\}$ and for any point $(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_s) \in \mathbf{Q}_1 \times \dots \times \mathbf{Q}_{j-1} \times \mathbf{Q}_{j+1} \times \dots \times \mathbf{Q}_s$ the function $f(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_s)$ extends holomorphically to $\hat{\mathbf{Q}}_j$. Therefore, any function from $\mathcal{F}$ may be treated as a separately holomorphic function on $\mathbb{X}_{s,1}((\mathbf{Q_j}, \hat{\mathbf{Q}}_j)_{j=1}^s)$, which finishes the proof. $\square$

Remark 3.11. Let $\mathbf{Q}$ be as in Theorem 3.8 and let $\hat{\mathbf{X}}$ be constructed via Theorem 3.8. We may consider the extension theorem for $\mathbf{Q}$ with analytic singularities given by the analytic subset of positive pure codimension of $\hat{\mathbf{X}}$.

Using the analogous argument to the one given in the proof of Theorem 2.12 from [9] we can state and prove the extension theorem for $\mathscr{A}$-crosses with analytic singularities, parallel to Theorem 2.12 (case $F = \varnothing$) therein.

4. "Nice" descriptions and some geometry

We know that in the context of (N,k)-crosses and generalized (N,k)-crosses, their envelopes of holomorphy have a nice description in terms of the relative extremal function of the set A_j with respect to D_j. For the $\mathscr{A}$-crosses the existence of such description is more subtle problem. For $N \in \{2,3\}$ the situation is simple, as in this case the $\mathscr{A}$-crosses which are interesting from the point of view of Theorem 3.8 (i.e. those which contain $\mathbb{X}_{2,1}((A_j,D_j)_{j=1,2}), \mathbb{X}_{3,1}((A_j,D_j)_{j=1,2,3})$, respectively) give nothing new in comparison with "old" crosses (see Example 3.4). However, already in the case $N = 4$ such descriptions can be various.

Example 4.1. Let $N = 4$. Let $D_j \subset \mathbb{C}^{n_j}$ be a hyperconvex domain and let $A_j \subset D_j$ be compact, locally pluriregular, and locally L-regular (see [11]), $j = 1,\dots,4$. Consider the $\mathscr{A}$-crosses interesting from the point of view of Theorem 3.8. A straightforward computation shows that we have to consider exactly nine nontrivial (that is, different from the $(4,k)-$crosses) cases here (up to permutations of variables):

Case 1.

$$\mathbf{Q}^1 := \mathbf{Q}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{matrix} (A_1 \times A_2 \times A_3 \times D_4)\cup \\ (A_1 \times D_2 \times D_3 \times A_4)\cup \\ (D_1 \times A_2 \times A_3 \times A_4). \end{matrix}$$

Then after some calculations we conclude that

$$\hat{\mathbf{Q}}^1 = \{(z_1,z_2,z_3,z_4) \in D_1 \times D_2 \times D_3 \times D_4 : h_{A_1,D_1}(z_1) + h_{A_4,D_4}(z_4) + \max\{h_{A_2,D_2}(z_2), h_{A_3,D_3}(z_3)\} < 1\}.$$

Case 2.

$$\mathbf{Q}^2 := \mathbf{Q}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{matrix} (A_1 \times A_2 \times A_3 \times D_4)\cup \\ (D_1 \times A_2 \times D_3 \times A_4)\cup \\ (D_1 \times D_2 \times A_3 \times A_4). \end{matrix}$$

Then

$$\hat{\mathbf{Q}}^2 = \{(z_1, z_2, z_3, z_4) \in D_1 \times D_2 \times D_3 \times D_4 : h_{A_4,D_4}(z_4) + \max\{h_{A_1,D_1}(z_1), h_{A_2,D_2}(z_2) + h_{A_3,D_3}(z_3)\} < 1\}.$$

Case 3.

$$\mathbf{Q}^3 := \mathbf{Q}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{array}{l} (A_1 \times A_2 \times A_3 \times D_4) \cup \\ (A_1 \times D_2 \times D_3 \times A_4) \cup \\ (D_1 \times A_2 \times D_3 \times A_4) \cup \\ (D_1 \times D_2 \times A_3 \times A_4). \end{array}$$

The envelope of holomorphy of $\mathbf{Q}^3$ is of the form

$$\{(z_1, z_2, z_3, z_4) \in \hat{\mathbb{X}}_{3,2}((A_j, D_j)_{j=1}^3) \times D_4 : h_{A_4,D_4}(z_4) + h_{A_1 \times A_2 \times A_3, \hat{\mathbb{X}}_{3,2}((A_j,D_j)_{j=1}^3)} < 1\}.$$

Using Proposition 4.4 below we see that here the desired description is given by

$$\hat{\mathbf{Q}}^3 = \Big\{(z_1, z_2, z_3, z_4) \in D_1 \times D_2 \times D_3 \times D_4 : h_{A_4,D_4}(z_4) + \max\Big\{\frac{1}{2}(h_{A_1,D_1}(z_1) + h_{A_2,D_2}(z_2) + h_{A_3,D_3}(z_3)), \max_{j=1,2,3}\{h_{A_j,D_j}(z_j)\}\Big\} < 1\Big\}.$$

Case 4.

$$\mathbf{Q}^4 := \mathbf{Q}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{l} (A_1 \times A_2 \times D_3 \times D_4) \cup \\ (A_1 \times D_2 \times A_3 \times D_4) \cup \\ (D_1 \times A_2 \times A_3 \times D_4) \cup \\ (D_1 \times A_2 \times D_3 \times A_4). \end{array}$$

Take a function $f \in \mathcal{F}$. Observe that:

- For any fixed point $a_2 \in A_2$, the function $f(\cdot, a_2, \cdot)$ is holomorphic on $\hat{\mathbb{X}}_{3,2}((A_j, D_j)_{j=1,3,4})$. Moreover, for any point $(z_1, z_3, z_4) \in A_1 \times A_3 \times D_4$, the function $f(z_1, \cdot, z_3, z_4)$ is holomorphic on D_2. Therefore, using classical cross theorem (see [1]), we conclude that the envelope of holomorphy of $\mathbf{Q}^4$ equals (up to permutation of variables)

$$\{(z_2, z_1, z_3, z_4) \in D_2 \times \hat{\mathbb{X}}_{3,2}((A_j, D_j)_{j=1,3,4}) : h_{A_2,D_2}(z_2) + h^{\star}_{A_1 \times A_3 \times D_4, \hat{\mathbb{X}}_{3,2}((A_j,D_j)_{j=1,3,4})}(z_1, z_3, z_4) < 1\}.$$

We shall prove the following

Claim.

$$h^\star_{A_1\times A_3\times D_4,\hat{\mathbb{X}}_{3,2}((A_j,D_j)_{j=1,3,4})}(z_1,z_3,z_4)=\\ \max\{h_{A_1,D_1}(z_1),h_{A_3,D_3}(z_3),h_{A_1,D_1}(z_1)+h_{A_3,D_3}(z_3)+h_{A_4,D_4}(z_4)-1\}$$

for $(z_1,z_3,z_4)\in\hat{\mathbb{X}}_{3,2}((A_j,D_j)_{j=1,3,4})$.

Proof of Claim. The inequality $\geq$ is obvious as the right-hand side is from the defining family for the left-hand side. In order to prove the equality, we may assume that D_j is relatively compact and strongly pseudoconvex, $j=1,\dots,4$ (use Proposition 3.2.25 from [7]).

Choose an increasing sequence $(K_j)_{j=1}^\infty$ of holomorphically convex locally L-regular compacta in D_4 containing A_4 and such that $\bigcup_{j=1}^\infty K_j=D_4$ (this is possible by the existence of an exhausting sequence of holomorphically convex compacta in D_4 and using [14]; see also [10]). Put

$$L_s(z_1,z_3,z_4):=h^\star_{A_1\times A_3\times K_s,\hat{\mathbb{X}}_{3,2}((A_j,D_j)_{j=1,3,4})}(z_1,z_3,z_4)$$

for $s\in\mathbb{N}$ and $(z_1,z_3,z_4)\in\hat{\mathbb{X}}_{3,2}((A_j,D_j)_{j=1,3,4})$. Observe that the functions L_s are continuous (use [11]). Moreover, we have the equality $(dd^cL_s)^n=0$ on the set $\hat{\mathbb{X}}_{3,2}((A_j,D_j)_{j=1,3,4})\setminus(A_1\times A_3\times K_s)=:V_s$, $s\in\mathbb{N}$, where $(dd^c)^n$ is the complex Monge-Ampère operator ([2]).

Furthermore, for any $z_0\in\partial V_s$ there is

$$\liminf_{V_s\ni z\to z_0}(\max\{h_{A_1,D_1}(z_1),h_{A_3,D_3}(z_3),\\ h_{A_1,D_1}(z_1)+h_{A_3,D_3}(z_3)+h_{A_4,D_4}(z_4)-1\}-L_s(z_1,z_3,z_4))\geq 0,$$

and, by the domination principle (Corollary 3.7.4 from [8]),

$$h^\star_{A_1\times A_3\times D_4,\hat{\mathbb{X}}_{3,2}((A_j,D_j)_{j=1,3,4})}(z_1,z_3,z_4)\leq L_s(z_1,z_3,z_4)=\\ \max\{h_{A_1,D_1}(z_1),h_{A_3,D_3}(z_3),h_{A_1,D_1}(z_1)+h_{A_3,D_3}(z_3)+h_{A_4,D_4}(z_4)-1\}$$

for $s\in\mathbb{N}$ and $(z_1,z_3,z_4)\in\hat{\mathbb{X}}_{3,2}((A_j,D_j)_{j=1,3,4})$, from which follows the conslusion. $\square$

Making use of the above claim we conclude that our description is given by

$$\hat{\mathbf{Q}}^4 = \{(z_1,z_2,z_3,z_4) \in D_1 \times D_2 \times D_3 \times D_4 : h_{A_2,D_2}(z_2)+$$
$$\max\{h_{A_1,D_1}(z_1), h_{A_3,D_3}(z_3), h_{A_1,D_1}(z_1)+h_{A_3,D_3}(z_3)+h_{A_4,D_4}(z_4)-1\} < 1\}.$$

Case 5.

$$\mathbf{Q}^5 := \mathbf{Q}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{array}{l} (A_1 \times A_2 \times D_3 \times D_4) \cup \\ (A_1 \times D_2 \times A_3 \times D_4) \cup \\ (A_1 \times D_2 \times D_3 \times A_4) \cup \\ (D_1 \times A_2 \times A_3 \times D_4) \cup \\ (D_1 \times A_2 \times D_3 \times A_4). \end{array}$$

Here we have

$$\hat{\mathbf{Q}}^5 = \{(z_1,z_2,z_3,z_4) \in D_1 \times D_2 \times D_3 \times D_4 : h_{A_1,D_1}(z_1) + h_{A_2,D_2}(z_2)+$$
$$\max\{h_{A_3,D_3}(z_3) + h_{A_4,D_4}(z_4) - 1, 0\} < 1\}.$$

Case 6.

$$\mathbf{Q}^6 := \mathbf{Q}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{array}{l} (A_1 \times A_2 \times D_3 \times D_4) \cup \\ (D_1 \times A_2 \times A_3 \times D_4) \cup \\ (D_1 \times D_2 \times A_3 \times A_4). \end{array}$$

Take a function $f \in \mathcal{F}$. Observe that:

• For any fixed point $a_2 \in A_2$ the function $f(\cdot, a_2, \cdot)$ is holomorphic on the set $(\widehat{(A_1 \times D_3) \cup (D_1 \times A_3)}) \times D_4$. Moreover, for any point $(z_1, z_3, z_4) \in D_1 \times A_3 \times A_4$, the function $f(z_1, \cdot, z_3, z_4)$ is holomorphic on D_2. Therefore, using classical cross theorem, we conclude that the envelope of holomorphy of $\mathbf{Q}^6$ equals (up to permutation of variables)

$$\{(z_2,z_1,z_3,z_4) \in D_2 \times (\widehat{(A_1 \times D_3) \cup (D_1 \times A_3)}) \times D_4 : h_{A_2,D_2}(z_2)$$
$$+ h^{\star}_{D_1\times A_3\times A_4, (\widehat{(A_1\times D_3)\cup(D_1\times A_3)})\times D_4}(z_1,z_3,z_4) < 1\}.$$

• On the other hand, for any point $(z_1, z_3) \in D_1 \times A_3$, the function $f(z_1, \cdot, z_3, \cdot)$ is holomorphic on $\widehat{(A_2 \times D_4) \cup (D_2 \times A_4)}$. Moreover, for any point $(z_2, z_4) \in A_2 \times D_4$, the function $f(\cdot, z_2, \cdot, z_4)$ is holomorphic on $\widehat{(A_1 \times D_3) \cup (D_1 \times A_3)}$.

Using once again classical cross theorem, we see that the envelope of holomorphy of $\mathbf{Q}$ equals (up to permutation of variables)

$$\{(z_2,z_4,z_1,z_3)\in \widehat{(A_2\times D_4)\cup(D_2\times A_4)}\times\widehat{(A_1\times D_3)\cup(D_1\times A_3)}:$$
$$h^\star_{D_1\times A_3,\widehat{(A_1\times D_3)\cup(D_1\times A_3)}}(z_1,z_3)+h^\star_{A_2\times D_4,\widehat{(A_2\times D_4)\cup(D_2\times A_4)}}(z_2,z_4)<1\}.$$

From the above bullets follows that any arbitrary point $(z_1,z_2,z_3,z_4)\in D_1\times D_2\times D_3\times D_4$ satisfies the following system of conditions:

$$h_{A_2,D_2}(z_2)+h_{A_4,D_4}(z_4)<1, \tag{4.1}$$

$$h_{A_1,D_1}(z_1)+h_{A_3,D_3}(z_3)<1, \tag{4.2}$$

$$h^\star_{D_1\times A_3,\widehat{(A_1\times D_3)\cup(D_1\times A_3)}}(z_1,z_3)+h^\star_{A_2\times D_4,\widehat{(A_2\times D_4)\cup(D_2\times A_4)}}(z_2,z_4)<1 \tag{4.3}$$

iff it satisfies the following system of conditions:

$$h_{A_2,D_2}(z_2)+h_{A_4,D_4}(z_4)<1, \tag{4.4}$$

$$h_{A_1,D_1}(z_1)+h_{A_3,D_3}(z_3)<1, \tag{4.5}$$

$$h_{A_2,D_2}(z_2)+h^\star_{D_1\times A_3,\widehat{(A_1\times D_3)\cup(D_1\times A_3)}}<1. \tag{4.6}$$

We shall prove the following

Claim.

$$h^\star_{A_2\times D_4,\widehat{(A_2\times D_4)\cup(D_2\times A_4)}}(z_2,z_4)=h_{A_2,D_2}(z_2)$$

for $(z_2,z_4)\in\widehat{(A_2\times D_4)\cup(D_2\times A_4)}$.

Proof of Claim. Observe that the inequality $\geq$ is evident. Suppose, seeking a contradiction, that there exists a point $(z_2^0,z_4^0)\in\widehat{(A_2\times D_4)\cup(D_2\times A_4)}$, and numbers $\alpha,\beta\in(0,1)$ such that

$$\alpha=h^\star_{A_2\times D_4,\widehat{(A_2\times D_4)\cup(D_2\times A_4)}}(z_2^0,z_4^0)>h_{A_2,D_2}(z_2^0)=\alpha-\beta.$$

We know (Proposition 4.5.2 from [8]) that there exists a $z_3^0\in D_3$ such that $h_{A_3,D_3}(z_3^0)=1-\alpha$.

Finally, take any $z_1^0\in A_1$. Then

$$1-\alpha=h_{A_3,D_3}(z_3^0)\leq h^\star_{D_1\times A_3,\widehat{(D_1\times A_3)\cup(A_1\times D_3)}}(z_1^0,z_3^0)$$
$$\leq h_{A_1\times A_3,\widehat{(D_1\times A_3)\cup(A_1\times D_3)}}(z_1^0,z_3^0)=h_{A_1,D_1}(z_1^0)+h_{A_3,D_3}(z_3^0)=1-\alpha.$$

Therefore, $h^\star_{D_1\times A_3,\widehat{(D_1\times A_3)\cup(A_1\times D_3)}}(z_1^0,z_3^0)=1-\alpha$. Observe that in this situation the point $(z_1^0,z_2^0,z_3^0,z_4^0)$ satisfies the conditions (4.1), (4.2), (4.4), (4.5), and (4.6), while it does not satisfy the condition (4.3), which is a contradiction. $\square$

Making use of the above claim we conclude that the description under our interest is:

$$\hat{\mathbf{Q}}^6 = \{(z_1, z_2, z_3, z_4) \in D_1 \times D_2 \times D_3 \times D_4 : \max\{h_{A_1,D_1}(z_1) + h_{A_3,D_3}(z_3), \\ h_{A_2,D_2}(z_2) + h_{A_4,D_4}(z_4), h_{A_2,D_2}(z_2) + h_{A_3,D_3}(z_3)\} < 1\}.$$

Case 7.

$$\mathbf{Q}^7 := \mathbf{Q}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{array}{l} (A_1 \times A_2 \times D_3 \times D_4) \cup \\ (A_1 \times D_2 \times D_3 \times A_4) \cup \\ (D_1 \times A_2 \times A_3 \times D_4) \cup \\ (D_1 \times D_2 \times A_3 \times A_4). \end{array}$$

Here we have

$$\hat{\mathbf{Q}}^7 = \{(z_1, z_2, z_3, z_4) \in D_1 \times D_2 \times D_3 \times D_4 : \\ \max\{h_{A_2,D_2}(z_2) + h_{A_4,D_4}(z_4), h_{A_1,D_1}(z_1) + h_{A_3,D_3}(z_3)\} < 1\}.$$

Case 8.

$$\mathbf{Q}^8 := \mathbf{Q}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{array}{l} (A_1 \times D_2 \times D_3 \times D_4) \cup \\ (D_1 \times A_2 \times A_3 \times D_4) \cup \\ (D_1 \times D_2 \times A_3 \times A_4). \end{array}$$

Here we have

$$\hat{\mathbf{Q}}^8 = \{(z_1, z_2, z_3, z_4) \in D_1 \times D_2 \times D_3 \times D_4 : h_{A_1,D_1}(z_1) + \\ \max\{h_{A_3,D_3}(z_3), \max\{h_{A_2,D_2}(z_2) + h_{A_4,D_4}(z_4) - 1, 0\}\} < 1\}.$$

Case 9.

$$\mathbf{Q}^9 := \mathbf{Q}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{array}{l} (A_1 \times D_2 \times D_3 \times D_4) \cup \\ (D_1 \times A_2 \times A_3 \times D_4) \cup \\ (D_1 \times A_2 \times D_3 \times A_4) \cup \\ (D_1 \times D_2 \times A_3 \times A_4). \end{array}$$

The envelope of holomorphy of $\mathbf{Q}^9$ is of the form

$$\{(z_1, z_2, z_3, z_4) \in D_1 \times D_2 \times D_3 \times D_4 : \\ h_{A_1,D_1}(z_1) + h^{\star}_{\mathbb{X}_{3,1}((A_j,D_j)_{j=1}^3),D_1 \times D_2 \times D_3,} < 1\}.$$

By Proposition 4.6 below we get the following description:

$$\hat{\mathbf{Q}}^9 = \{(z_1,z_2,z_3,z_4) \in D_1 \times D_2 \times D_3 \times D_4 : h_{A_1,D_1}(z_1)+ \frac{1}{2}(h_{A_2,D_2}(z_2) + h_{A_3,D_3}(z_3) + h_{A_4,D_4}(z_4) - 1) < 1\}.$$

After the above example we could possibly expect that any set which can be described in a similar way as all envelopes of holomorphy from the above example, must be an envelope of holomorphy of certain $\mathscr{A}$-cross. This is however not the case. For instance, let $N = 4$ and keep the assumptions from Example 4.1. Put

$$\tilde{\mathbf{Q}} := \{(z_1,z_2,z_3,z_4) \in D_1 \times D_2 \times D_3 \times D_4 : \max\{h_{A_2,D_2}(z_2) + h_{A_4,D_4}(z_4), h_{A_1,D_1}(z_1)+h_{A_2,D_2}(z_2)+h_{A_3,D_3}(z_3), h_{A_1,D_1}(z_1)+h_{A_3,D_3}(z_3)+h_{A_4,D_4}(z_4)\} < 1\}.$$

It follows from some simple but a little bit tedious calculations that there is no $\mathscr{A}$-cross $\mathbf{Q}$ with the defining matrix of dimension $(l_1 + l_2 + l_3 + l_4) \times 4$ such that the set $\tilde{\mathbf{Q}}$ is the envelope of holomorphy of $\mathbf{Q}$.

As the following two examples show (and so the previous one), if we consider "thin" $\mathscr{A}$-cross (in the sense that there are a few of D_j's in each of its branches) we should expect rather "small" envelope of holomorphy of such object, while in the situation where our $\mathscr{A}$-cross is "fat" (a lot of D_j's in each of its branches), then we obtain quite "big" envelope of holomorphy.

Example 4.2 (cf. Example 4.1, Case 2)**.** Let $N > 3$ be arbitrary. Keep the assumptions of Theorem 3.8. Consider the $\mathscr{A}$-cross $\mathbf{Q}' := \mathbf{Q}(\mathscr{A})$, where the matrix $\mathscr{A} = (\alpha_j^i)_{j=1,\dots,N, i=1,\dots N-1}$ is as follows

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 1 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$\hat{\mathbf{Q}}' = \{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : h_{A_N,D_N}(z_N) + \max\{h_{A_{N-1},D_{N-1}}(z_{N-1}), \sum_{j=1}^{N-2} h_{A_j,D_j}(z_j)\} < 1\}.$$

We see that the envelope of holomorphy of $\mathbf{Q}'$ is in general essentially contained in the set $\hat{\mathbb{X}}_{N,2}((A_j, D_j)_{j=1}^N)$, no matter how large the N is.

Example 4.3 (cf. Example 4.1, Case 8)**.** Let $N > 3$ be arbitrary. Keep the assumptions of Theorem 3.8. Consider the $\mathscr{A}$-cross $\mathbf{Q}'' := \mathbf{Q}(\mathscr{A})$, where the matrix $\mathscr{A} = (\alpha_j^i)_{j=1,\dots,N, i=1,\dots N-1}$ is as follows

$$\begin{pmatrix} 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & 1 & \cdots & 0 \\ 1 & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Then

$$\hat{\mathbf{Q}}'' = \{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : h_{A_2,D_2}(z_2) + \max\{h_{A_1,D_1}(z_1), \max\{\sum_{j=3}^{N} h_{A_j,D_j}(z_j) - N + 3, 0\}\} < 1\}.$$

Observe that the above set contains

$$(\hat{\mathbb{X}}_{2,1}((A_j, D_j)_{j=1}^2) \times D_3 \times \dots \times D_N) \cap \hat{\mathbb{X}}_{N,N-2}((A_j, D_j)_{j=1}^N).$$

Proposition 4.4. *Let $D_j \subset \mathbb{C}^{n_j}$ be a hyperconvex domain and let $A_j \subset D_j$ be locally pluriregular, compact and locally L-regular, $j = 1, \dots, N$. Put $\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N)$. Then*

$$L(z) := h_{A_1 \times \dots \times A_N, \hat{\mathbf{X}}_{N,k}}(z) = \max\left\{\frac{1}{k}\sum_{j=1}^{N} h_{A_j,D_j}(z_j), \max_{j=1,\dots,N}\{h_{A_j,D_j}(z_j)\}\right\} =: R(z)$$

for every $z = (z_1, \dots, z_N) \in \hat{\mathbf{X}}_{N,k}$.

Proof. Note that in view of the assumptions h_{A_j,D_j} is continuous, $j = 1, \dots, N$, and so is L (the last observation follows from [11]).

Observe that the inequality $L \geq R$ is obvious and R is from the defining family for L, so we only need to prove the opposite inequality.

We may assume that D_j is relatively compact and strongly pseudoconvex, $j = 1, \dots, N$.

We have $(dd^c L)^n = 0$ on the set $\hat{\mathbf{X}}_{N,k} \setminus (A_1 \times \ldots \times A_N) =: V$. Moreover, for any $z_0 \in \partial V$ there is

$$\liminf_{V \ni z \to z_0} (R(z) - L(z)) \geq 0,$$

from which follows (in view of the domination principle)

$$R \geq L \text{ on } \hat{\mathbf{X}}_{N,k} \setminus (A_1 \times \ldots \times A_N).$$

Thus $R \geq L$ on $\hat{\mathbf{X}}_{N,k}$. $\square$

Example 4.5. Let $0 < r < R$ be two real numbers. Let $D_j = \mathbb{B}_{n_j}(R) \subset \mathbb{C}^{n_j}$ be the euclidean open ball with center at 0 and radius R and let $A_j = \overline{\mathbb{B}_{n_j}}(r)$ be the euclidean closed ball with center at 0 and radius r, $j = 1, \dots, N$. Then $h_{A_j,D_j} = \max\left\{0, \frac{\log \frac{||\cdot||}{r}}{\log \frac{R}{r}}\right\}$, $j = 1, \dots, N$, and thus

$$h_{A_1 \times \ldots \times A_N, B}(z) = \max\left\{\frac{1}{k}\sum_{j=1}^{N} \max\left\{0, \frac{\log \frac{||z_j||}{r}}{\log \frac{R}{r}}\right\}, \max_{j=1,\dots,N}\left\{0, \frac{\log \frac{||z_j||}{r}}{\log \frac{R}{r}}\right\}\right\},$$

for $z = (z_1, \dots, z_N) \in B$, where

$$B := \left\{(z_1, \dots, z_N) \in D_1 \times \ldots \times D_N : \sum_{j=1}^{N} \max\left\{0, \frac{\log \frac{||z_j||}{r}}{\log \frac{R}{r}}\right\} < k\right\}.$$

Proposition 4.6. *Let $1 \leq k < l \leq N$. Let $D_j \subset \mathbb{C}^{n_j}$ be a hyperconvex domain and let $A_j \subset D_j$ be compact, locally pluriregular and locally L-regular, $j = 1, \dots, N$. Then*

$$L(z) := h^{\star}_{\hat{\mathbf{X}}_{N,k}, \hat{\mathbf{X}}_{N,l}}(z) = h^{\star}_{\mathbf{X}_{N,k}, \hat{\mathbf{X}}_{N,l}}(z) = \max\left\{0, \frac{\sum\limits_{j=1}^{N} h_{A_j,D_j}(z_j) - k}{l-k}\right\} =: R(z)$$

for $z = (z_1, \dots, z_N) \in \hat{\mathbf{X}}_{N,l}$.

Proof. Observe that we only need to prove the third equality. We may assume that D_j is relatively compact and strongly pseudoconvex, $j = 1, \dots, N$.

Observe that the inequality $\geq$ is obviuos, as the function R belongs to the

defining family for L. All we need to show is the opposite inequality.
For any $j = 1, \ldots, N$ choose an increasing sequence $(K_j^s)_{s\in\mathbb{N}}$ of holomorphically convex locally L-regular compacta in D_j containing A_j and such that $\bigcup_{s=1}^{\infty} K_j^s = D_j$ (this is possible by the existence of an exhausting sequence of holomorphically convex compacta in D_j and using [14]).

Define

$$L_s(z) := h_{\mathbf{X}^s_{N,k}, \hat{\mathbf{X}}_{N,l}}(z),$$

$z \in \hat{\mathbf{X}}_{N,l}$, where $\mathbf{X}^s_{N,k} := \mathbb{X}_{N,k}((A_j, K_j^s)_{j=1}^N)$ (the (N,k)-crosses are defined for open D_j's. However, in our context, the definition of $\mathbf{X}^s_{N,k}$ - formally the same as the definition of the (N,k)-cross - makes sense, as it is only set-theoretical).

Notice that the functions L_s are all continuous (see [11]).
For a fixed $s \in \mathbb{N}$ there is

$$L_s(z) \geq \max\left\{0, \frac{\sum_{j=1}^{N} h_{A_j,D_j}(z_j) - k}{l-k}\right\}.$$

Furthermore, $(dd^c h_{\mathbf{X}^s_{N,k}, \hat{\mathbf{X}}_{N,l}})^n = 0$ on $V_s := \hat{\mathbf{X}}_{N,l} \setminus \mathbf{X}^s_{N,k}$ and for any $z_0 \in \partial V_s$ there is

$$\liminf_{V_s \ni z \to z_0} (R(z) - L_s(z)) \geq 0,$$

from which follows that $R \equiv L_s$, $s \in \mathbb{N}$, and so $R \equiv L$. $\square$

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