CHARACTERIZATIONS OF BOUNDARY PLURIPOLAR HULLS

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Abstract. We present some basic properties of the so called boundary relative extremal function and discuss *boundary pluripolar* sets and *boundary pluripolar hulls.* We show that for B-regular domains the boundary pluripolar hull is always trivial on the boundary of the domain and present a "boundary version" of Zeriahi's theorem on the completeness of pluripolar sets.

1. Introduction

Throughout the paper D will denote a bounded domain in $\mathbb{C}^n$, $\mathrm{PSH}(D)$ the family of all plurisubharmonic functions on D and A a subset in the boundary of D. For any function $u : D \to \mathbb{R} \cup \{-\infty\}$ and $x \in \overline{D}$ set

$$u^*(x) = \limsup_{z\to x, z\in D} u(z) = \lim_{r\to 0} \sup_{B(x,r)\cap D} u,$$

the upper semicontinuous regularization of u on $\overline{D}$. We let $\mathbb{D}$ be the unit disc, $\mathbb{T}$ the unit circle and $\mathbb{B}$ the unit ball in $\mathbb{C}^2$.

Siciak, cf. [11] introduced the *relative extremal function* ω^* where ω is defined as follows. Given an open set D in $\mathbb{C}^n$ and a compact subset E of D

$$\omega(z, E, D) = \sup\{u(z); u \in \mathrm{PSH}(D), u \le 0, u \le -1 \text{ on } E\} \quad , z \in D.$$

Siciak's definition makes sense for nonempty subsets A of ∂D. For $z \in D$ one defines, cf. [9, 8, 4],

$$\omega(z, A, D) = \sup\{u(z) : u \in \mathrm{PSH}(D), u < 0, u^* \le -1 \text{ on } A\}.$$

If A is empty we set $\omega(., A, D) \equiv 0$. We will call $\omega^*(., A, D)$ the *boundary relative extremal function.* It is a special case of the (regularization of) the Perron- Bremermann function hence is always maximal in D. For a bounded function f on ∂D Perron-Bremermann function u_f is defined as

$$u_f = \sup\{v \in \mathrm{PSH}(D), v^* \le f \text{ on } \partial D\}.$$

In Section 2 we will study $\omega(., A, D)$ somewhat further and give some additional properties and applications of it.

Following Sibony, cf. [10], we will say that a bounded domain $D \subset \mathbb{C}^n$ is *B-regular* if every $f \in C(\partial D)$ can be extended to a plurisubharmonic function in D that is continuous on $\overline{D}$. In [10] it is proved that the following statements are equivalent:

- D is B-regular;

2010 *Mathematics Subject Classification.* 32U05.

Key words and phrases. Plurisubharmonic functions, Pluripotential theory, Pluripolar sets B-regular domains.

- For $z \in \partial D$ there is $u \in \mathrm{PSH}(D) \cap C(\overline{D})$ such that $u(z) = 0$ and $u < 0$ on $\overline{D} \setminus \{z\}$;
- There is $u \in \mathrm{PSH}(D) \cap C(\overline{D})$ such that $\lim_{z \to \partial D} u(z) = 0$ and $z \mapsto u(z) - |z|^2 \in \mathrm{PSH}(D)$.

For a B-regular domain D if $f \in C(\partial D)$ then $u_f \in \mathrm{PSH}(D) \cap C(\overline{D})$ and $u_f = f$ on ∂D, cf. [2].

For $A \subset \partial D$, it can happen that any $u \in \mathrm{PSH}(D)$ such that $u^*|_A = -\infty$ assumes the value $-\infty$ automatically on a bigger set in $\overline{D}$. For instance, set $\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2, |z_1^2| + |z_2^2| < 1\}$. Let $A_1 \subset \mathbb{T}$ be the closure of a half-circle. Set $A = A_1 \times \{0\}$. Any $u \in \mathrm{PSH}(\mathbb{B})$ such that $u^* \equiv -\infty$ on A is identically $-\infty$ in $\{z \in \mathbb{C}, |z| < 1\} \times \{0\}$. The phenomenon is similar to the occurrence of pluripolar hull.

We will call a subset $A \subset \partial D$ *b-pluripolar* (boundary pluripolar) if there exists a $u \in \mathrm{PSH}(D)$, $u \leq 0$, $u \not\equiv -\infty$, such that $A \subset \{u^* = -\infty\}$ and we will call a subset $A \subset \partial D$ *completely b-pluripolar* if there exists a $u \in \mathrm{PSH}(D)$, $u < 0$, $u \not\equiv -\infty$, such that $\{z \in \partial D, \quad u^*(z) = -\infty\} = A$. Zeriahi showed in [14] that if $E \subset D$ is pluripolar and an F_σ as well as a G_δ, then E is completely pluripolar in D i.e there exists $u \in PSH(D)$ with $E = \{z \in D : u(z) = -\infty\}$ if and only if E coincides with its pluripolar hull. We will define the *boundary pluripolar hull* in Definition 3.3 and employ $\omega(., A, D)$ to describe this in Section 3 and 4. We will show that for B-regular domains the b-pluripolar hull $\hat{A} \subset \overline{D}$ of a b-pluripolar set A is contained in $A \cup D$. It is perhaps mildly surprizing that no hull is picked up at the boundary. In particular we have Corollary 4.5 that for B-regular domains every b-pluripolar set that is simultaneously F_σ and G_δ is completely b-pluripolar.

In his thesis, [12] Wikström considered the function $V \in \mathrm{PSH}(\mathbb{B})$:

$$V(z) = \log \frac{|z_2|^2}{1 - |z_1|^2}$$

and observed that $V|_{\{z_2=0\}} = -\infty$ inside $\mathbb{B}$, but $V^*(z_1, 0) = 0$ for $|z_1| = 1$, cf. [13], Example 5.5. This example suggested to us that something like Corollary 4.5 could hold.

Acknowledgement. The first author is supported by the international PhD programme "Geometry and Topology in Physical Models" of the Foundation for Polish Science and he wishes to thank Professor Armen Edigarian for his help to accomplish this work.

2. Properties of ω

In this section we collect some elementary properties of ω.

Proposition 2.1. *If $A_1 \subset A_2 \subset \partial D$, then*

$$\omega(., A_2, D) \leq \omega(., A_1, D).$$

If $D_1 \subset D_2$ and $A \subset \partial D_1 \cap \partial D_2$, then on D_1 we have

$$\omega(., A, D_2) \leq \omega(., A, D_1).$$

Proposition 2.2. *Let $D \subset \mathbb{C}^n$ be B-regular and $A \subset \partial D$. Then $\omega^*(., A, D) = 0$ on $\partial D \setminus (\overline{A})^o$.*

Proof. If $\partial D \setminus (\overline{A})^o$ is empty there is nothing to prove, if not let $x \in \partial D \setminus (\overline{A})^o$ and $r > 0$. Let $z \in B(x,r) \cap \partial D \setminus \overline{A}$ and let U be a neighborhood of $\overline{A}$ that does not contain z. Then there exists $f \in C(\partial D, [-1, 0])$ such that $f = -1$ on A and $f = 0$ on $\partial D \setminus U$. As D is B-regular and $f \in C(\partial D)$ then $u_f \in PSH(D) \cap C(\overline{D})$ and $u_f = f$ on ∂D. We have $u_f \leq \omega(., A, D)$. Thus

$$0 = u_f(z) \leq \sup_{B(x,r)\cap D} u_f \leq \sup_{B(x,r)\cap D} \omega(., A, D) \leq 0.$$

This holds for all $r > 0$. Hence $0 = \lim_{r\to 0} \sup_{B(x,r)\cap D} \omega(., A, D) = \omega^*(x, A, D)$. □

Proposition 2.3. *Let $D \subset \mathbb{C}^n$ be B-regular and $A \subset \partial D$. Then for all x in the interior of A we have*

$$\lim_{y\to x, y\in D} \omega(y, A, D) = -1.$$

Proof. Let $x \in A$ be an interior point of A. Take $0 < r < \operatorname{dist}(x, \partial A)$. Let $f \in C(\partial D, [-1, 0])$ such that $f = -1$ on $B(x, r/2) \cap \partial D$ and $f = 0$ on $\partial D \setminus B(x, r)$. As D is B-regular then $u_f \in \mathrm{PSH}(D) \cap C(\overline{D})$ and $u_f = f$ on ∂D. Observe that for all negative $v \in \mathrm{PSH}(D)$ with $v^*|_A \leq -1$, one has $v \leq u_f$ hence $\omega(., A, D) \leq u_f$. Thus

$$-1 \leq \liminf_{y\to x\ y\in D} \omega(y, A, D) \leq \limsup_{y\to x\ y\in D} \omega(y, A, D) \leq \limsup_{y\to x} u_f(y) = u_f(x) = -1.$$

□

Corollary 2.4. *If $D \subset \mathbb{C}^n$ is a B-regular domain and $A \subset \partial D$ is open, then $\omega(., A, D) \in PSH(D)$ and it coincides with $\omega^*(., A, D)$.*

Proof. We know that $\omega(., A, D) \leq \omega^*(., A, D)$ on D. As $\omega(., A, D)$ is bounded, $\omega^*(., A, D)$ belongs to $\mathrm{PSH}(D)$ and is negative. From Proposition 2.3 we have $\omega^*(., A, D) \leq -1$ on A. Hence $\omega^*(., A, D) \leq \omega(., A, D)$ on D. □

Proposition 2.5. *Let D be a B-regular domain in $\mathbb{C}^n$ and $A \subset \partial D$ be open. Suppose that $\{D_j\}$ is an increasing sequence of B-regular domains in D such that $D = \cup D_j$ and $A \subset \cap_j \partial D_j$. Then*

$$\lim_{j\to\infty} \omega(x, A, D_j) = \omega(x, A, D), \qquad \textit{for } x \in D.$$

Proof. Set $v = \lim \omega(., A, D_j)$. By Proposition 2.1, $\omega(., A, D_{j+1}) \leq \omega(., A, D_j)$ and by Corollary 2.4, $\omega(., A, D_j) \in \mathrm{PSH}(D_j)$, hence $v \geq \omega(., A, D)$. Now $v \in \mathrm{PSH}(D)$ and $v^* \leq -1$ on A, therefore $v \leq \omega(., A, D)$. It follows that $v = \omega(., A, D)$. □

Problem 2.6. Can the condition that A is open, be dropped?

Proposition 2.7. *For $D \subset \mathbb{C}^n$ a B-regular domain and $A \subset \partial D$ we have*

$$\omega(., A, D) = \sup_{A\subset V,\ V \textit{ open}} \omega(., V, D).$$

Corollary 2.8. *Let $D \subset \mathbb{C}^n$ be a B-regular domain and $(A_j)_j$ be a decreasing sequence of open sets in ∂D. Then $(\omega(., A_j, D))_j$ increases to $\omega(z, A, D)$ where $A = \cap A_j$.*

Proposition 2.9. *Assume that $A_1 \subset A_2 \subset \cdots \subset \partial D$ are open sets. Put $A = \cup A_j$. Then*

$$\lim_{j\to\infty} \omega(z, A_j, D) = \omega(z, A, D), \quad z \in D.$$

Proof. As A is open then $\omega(.,A,D) \in PSH(D)$ Corollary 2.4. Set $u(z) = \lim_{j\to\infty} \omega(z, A_j, D)$ for $z \in D$. Note that the sequence is decreasing, so $u \in PSH(D)$ and $u \geq \omega(.,A,D)$. On the other hand, $u \leq \omega(.,A_j,D)$ means that $u^* \leq -1$ on all A_j hence $u^* \leq -1$ on A. That means u is in the family defining $\omega(.,A,D)$. □

Proposition 2.10. *Let $D \subset \mathbb{C}^n$ be a domain and $A_j \subset \partial D$ be an increasing sequence of compact sets then*

$$\lim_{j\to\infty} \omega(.,A_j,D) = \omega(.,A,D)$$

where $A = \cup A_j$.

Proof. It is clear that $\omega(.,A,D) \leq \lim \omega(.,A_j,D)$. Let $\epsilon > 0$, $x \in D$ then for all $j > 0$ there is $u_j \in PSH(D)$ negative such that $\omega(x,A_j,D) \leq u_j(x) + \epsilon$. Set $V_j = \{u_j^* < -1+\epsilon\} \cap \partial D$, recall that $u_j \leq \omega(.,V_j,D) + \epsilon$. We get an open neighborhood V of A on setting $V = \cup_j V_j$. By Proposition 2.9 and Proposition 2.1 one has

$$\omega(x,A,D) \leq \lim_j \omega(x,A_j,D) \leq \lim_{j\to\infty} \omega(x,V_j,D) = \omega(x,V,D) + \epsilon \leq \omega(x,A,D) + \epsilon.$$

This for all $x \in D$ and $\epsilon > 0$.

□

Proposition 2.11. *Let D be an open set in $\mathbb{C}^n$, and let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be a sequence of compact subsets of ∂D. Then at each point in D*

$$\lim_{j\to\infty} \omega(.,A_j,D) = \omega(.,A,D),$$

where $A = \cap_{j=1}^{\infty} A_j$.

Proof. Clearly, $\omega(.,A_1,D) \leq \omega(.,A_2,D) \leq \ldots$ hence the limit exists. Take a negative function $v \in \mathrm{PSH}(D)$ such that $v^*|A \leq -1$. As the set $V = \{z \in D : v(z) - \epsilon < -1\}$ is open and A is compact, we can find an open set U containing A such that $U \cap D \subset V$. There exists j_0 such that for each $j \geq j_0$, $A_j \subset U$. Therefore $v - \epsilon \leq \omega(.,A_j,D)$ for $j \geq j_0$. As a consequence, $v - \epsilon \leq \lim_{j\to\infty} \omega(.,A_j,D)$, and so $\omega(.,A,D) - \epsilon \leq \lim_{j\to\infty} \omega(.,A_j,D)$, this for all $\epsilon > 0$. The opposite inequality is trivial. □

Now we would like to know whether one can define ω on $PHS(D) \cap C(\overline{D})$. Consider the function

$$\widetilde{\omega}(z,A,D) = \sup\{u(z); u \in PSH(D) \cap C(\overline{D}); u \leq 0; u|A \leq -1\}.$$

For all $A \subset \partial D$ we have $\widetilde{\omega}(.,A,D) \leq \omega(.,A,D)$ and $\tilde{\omega}(.,A,D) = \tilde{\omega}(.,\overline{A},D)$.

Proposition 2.12. *Let $D \subset \mathbb{C}^n$ be B-regular, $A \subset \partial D$ then*

$$\limsup_{y\to x} \widetilde{\omega}(y,A,D) = 0 \text{ for all } x \in \overline{\partial D \setminus A},$$

If x is an interior point of A then

$$\lim_{y\to x} \widetilde{\omega}(y,A,D) = -1.$$

Remark that for any $A \subset \partial D$ the plurisubharmonic function $\widetilde{\omega}^*(.,A,D)$ does not belong to the family defining $\widetilde{\omega}(.,A,D)$. Proposition 2.13 is connected to Problem 27.4 in [9].

Proposition 2.13. *Let D be a B-regular domain in $\mathbb{C}^n$ and $A \subset \partial D$ be closed, then*

$$\widetilde{\omega}(.,A,D) = \omega(.,A,D).$$

Proof. It is clear that $\tilde{\omega}(.,A,D) \leq \omega(.,A,D)$. Let $\epsilon > 0$ and u in the family defining $\omega(.,A,D)$ then by Wikstrom and Dini's theorems there is $v \in PSH(D) \cap C(\overline{D})$ negative such that $u^* \leq v < -1+\epsilon$ on A that means $u \leq v \leq \tilde{\omega}(.,A,D)+\epsilon$. This for all u hence $\omega(.,A,D) \leq \tilde{\omega}(.,A,D)+\epsilon$ for all ϵ. □

Remark 2.14. There is no hope to get $\omega(.,A,D) = \tilde{\omega}(.,A,D)$ for every $A \subset \partial D$. For instance if A is countable and dense in ∂D we get $\omega(.,A,D) = 0$ almost everywhere while $\tilde{\omega}(.,A,D) \equiv -1$.

Here we look at the link between the boundary relative extremal function and the relative extremal function "in usual sense". Let $j > 0$, $A \subset \partial D$ non dense. Set $E_j = \{z \in \mathbb{C}^n, d(z,A) < 1/j\} \cap D$. Consider the function

$$u_{E_j,D}(z) = \sup\{u(z) : u \in PSH(D), u < 0, u|E_j \leq -1\}.$$

Proposition 2.15. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $A \subset \partial D$ be closed then*

$$\lim_{j\to\infty} u_{E_j,D} = \omega(.,A,D).$$

Proof. For all $j > 0$, we have $u_{E_j,D} \in PSH(D)$, $u_{E_j,D} < 0$, $\limsup u_{E_j,D} \leq -1$ on A then $u_{E_j,D} \leq \omega(.,A,D)$ as consequence $\lim_{j\to\infty} u_{E_j,D} \leq \omega(.,A,D)$ Let u be in the family defining $\omega(.,A,D)$ and $\epsilon > 0$ there is an open set $U \subset \mathbb{C}^n$ containing A such that $u^* - \epsilon < -1$ on $U \cap \overline{D}$ take $j > 1$ enough big such that $E_j \subset U \cap D$ then $u - \epsilon \leq u_{E_j,D} \leq \lim_{j\to\infty} u_{E_j,D}$. Hence $\omega(.,A,D) - \epsilon \leq \lim_{j\to\infty} u_{E_j,D}$ for all $\epsilon > 0$. □

Corollary 2.16. *Let $A \subset \partial D$ then there is a sequence of open set $E_j \subset D$ such that*

$$\lim_{j\to\infty} u_{E_j,D} = \omega(.,A,D), \text{ almost everywhere.}$$

3. Boundary pluripolar sets and boundary pluripolar hulls

As in the classical case the boundary relative extremal function can be used to describe boundary pluripolar sets. The characterizations of Sadullaev [9], Levenberg-Poletsky [7], also cf. [4], of pluripolar hulls and their proof also hold for b-pluripolar sets. We will include this result with its very similar proof for convenience of the reader in Proposition 3.5. As in the classical case a countable union of b-pluripolar set is b-pluripolar (Proposition 3.6). However, in contrast with the classical case where the relative extremal function $\omega^*(.,E,D)$ of a subset $E \subset D$ has the property that $\{z \in E, \omega^*(z,E,D) > -1\}$ is pluripolar, the set $\{z \in A, \omega^*(z,A,D) > -1\}$ is not in general b-pluripolar and the behavior of $\omega^*(z,A,D)$ at the boundary of D is not very informative, see Example 3.4.

Definition 3.1. We say that a subset $A \in \partial D$ is a *b-pluripolar set* if there exists a $u \in \mathrm{PSH}(D)$, $u \leq 0$, $u \not\equiv -\infty$, such that $u^* = -\infty$ on A.

It is well known that a compact set $K \subset \mathbb{T}$ in the boundary of the unit disc $\mathbb{D}$ is b-polar if and only if it has arc length 0, and that not all such sets are polar. Hence there exist b-polar sets that are not polar. This example can be modified to the several variables situation.

Example 3.2. Let K be a b-polar set in $\mathbb{T}$ that is not polar and let u be a subharmonic function on D such that $u \leq 0$ and $u^*|_K = -\infty$. Consider the function v on the unit ball $\mathbb{B} \subset \mathbb{C}^2$ defined by $v(z,w) = u(z^2+w^2)$. Let

$$A = \{(z,w) \in \partial\mathbb{B} : z^2 + w^2 \in K\}.$$

Then $v^* = -\infty$ on A, hence A is b-pluripolar. Now if A would be pluripolar we could find, invoking Josefson's theorem, cf. [5], $f \in \mathrm{PSH}(\mathbb{C}^2)$ so that $f|_A = -\infty$. Consider for $\alpha \in [0, 2\pi)$ the function f_α on $\mathbb{C}$ defined by $f_\alpha(\zeta) = f(\zeta\cos\alpha, \zeta\sin\alpha)$. It is subharmonic or identically equal to $-\infty$. Take a branch $h(z)$ of $\sqrt{z}$ with branch cut not meeting K. Then $f_\alpha \circ h = -\infty$ on K. It follows that $f_\alpha \equiv -\infty$. In particular $f = -\infty$ on $\mathbb{R}^2 \subset \mathbb{C}^2$, which is not a pluripolar set. The conclusion is that A is not pluripolar.

Definition 3.3. Let $A \subset \partial D$ be b-pluripolar. The set

$$\{z \in \overline{D} :\ u^*(z) = -\infty,\ \text{for all } u \in \mathrm{PSH}(D) \text{ with } u \not\equiv -\infty,\ u < 0,\ u^*|_A = -\infty\}$$

will be called the *b-pluripolar hull* of A and will be denoted by $\hat{A}$.

Example 3.4. Let $\mathbb{B}$ be the unit ball $B(0,1)$ and $A = A_\alpha = \{(e^{i\phi}\cos\alpha, e^{i\psi}\sin\alpha) : \phi, \psi \in [0, 2\pi)\}$, the distinguished boundary of a polydisc Δ_α contained in $\mathbb{B}$. We have $\omega^*(., A, \mathbb{B}) \equiv 0$ on $\partial\mathbb{B}$, see Proposition 2.2. But every $u \in \mathrm{PSH}(\mathbb{B})$ such that $u^*|A \equiv -\infty$ is identically $-\infty$ on the polydisc, hence $u \equiv -\infty$ on $\mathbb{B}$ and A is not b-pluripolar..

Similarly, for $E_m = \cup_{j=1}^m A_{\alpha_j}$, we also find $\omega^*(., E_m, \mathbb{B}) \equiv 0$ on $\partial\mathbb{B}$. However, if we choose $(\alpha_j)_j$ a dense sequence in $(0, 2\pi)$ we find for $z \in \partial\mathbb{B}$

$$0 = \lim_{m\to\infty} \omega^*(z, E_m, \mathbb{B}) \neq \omega^*(z, \lim_{m\to\infty} E_m, \mathbb{B}) = -1.$$

Indeed, if $u \in \mathrm{PSH}(D)$ is negative and $u^* \leq -1$ on all E_m we have $u \leq -1$ on $\cup_j \Delta_{\alpha_j} = \mathbb{B}$.

Proposition 3.5 (cf. [9, 7, 4])**.** *Let $D \subset \mathbb{C}^n$ be a domain in $\mathbb{C}^n$ and $A \subset \partial D$. Then the following conditions are equivalent :*

(1) $\omega^*(., A, D) \equiv 0$;
(2) *A is b-pluripolar.*

In this case

$$\hat{A} \cap D = \{z \in D, \quad \omega(., A, D) < 0\}.$$

Proof. If A is b-pluripolar, take any $v \in \mathrm{PSH}(D)$, $v < 0$, $v \not\equiv -\infty$ such that $v^* = -\infty$ on A. Then $\epsilon v \leq \omega(., A, D)$ for all $\epsilon > 0$, hence if for some z $\omega(z, A, D) < 0$ then $v(z) = -\infty$, and it follows that $z \in \hat{A}$. Moreover, for all z such that $v(z) > -\infty$ we find $\omega(z, A, D) = 0$, hence $\omega^*(., A, D) \equiv 0$.
Assume now that $\omega^*(., A, D) \equiv 0$. Let $z \in D$ be such that $\omega(z, A, D) = 0$. For $j \in \mathbb{N}$ there is a negative $u_j \in \mathrm{PSH}(D)$ with $u_j^*|_A \leq -1$ and $u_j(z) > -2^{-j}$. Define

$$v(y) = \sum_{j=1}^{\infty} u_j(y) \quad (y \in D).$$

Observe that $v(z) > -1$, hence as a limit of a decreasing sequence of negative plurisubharmonic functions, $v \in \mathrm{PSH}(D)$, negative and not identically $-\infty$. Moreover, $v^*|A \equiv -\infty$. We conclude that A is b-pluripolar and $z \notin \hat{A}$. □

Proposition 3.6. *Let D be a bounded domain in $\mathbb{C}^n$. Suppose that $A = \cup_j A_j$, where $A_j \subset \partial D$ for $j = 1, 2, \cdots$. If $\omega^*(., A_j, D) \equiv 0$ for each j, then $\omega^*(., A, D) \equiv 0$.*

Proof. By Proposition 3.5 above, we can choose $v_j \in PSH(D)$ such that $v_j < 0$ on D and $v_j^*|A_j \equiv -\infty$. Take a point $a \in (D \setminus \cup_j v_j^{*-1}(\{-\infty\}))$. By multiplying each of the functions v_j^* by a suitable positive constant, we may suppose that $v_j(a) > -2^{-j}$. As in the proof of the proposition above we check that $v = \sum_j v_j \in PSH(D)$, $v < 0$ on D , $v \not\equiv -\infty$ on D and $v^* = -\infty$ on A. By the previous proposition , $\omega^*(., A, D) \equiv 0$. □

Problem 3.7. Let D_1 and D_2 be B-regular domains in $\mathbb{C}^n$ with $D_1 \subset D_2$ and $A \subset \partial D_1 \cap \partial D_2$ be b-pluripolar for D_1. Is A b-pluripolar for D_2?

The problem above can be seen as the boundary version of Lelong first problem. An positive answer will be the boundary version of Josefson theorem.

The result below illustrates the role of the boundary relative extremal function in computing the boundary pluripolar hull in the boundary. For a complete characterization of the hull see Proposition 3.10 and Theorem 3.12.

Proposition 3.8. *Let $D \subset \mathbb{C}^n$ be a B-regular domain and $A \subset \partial D$ be b-pluripolar. Then $A \subset \hat{A} \subset \overline{A}$.*

Proof. Let $z \in \partial D \setminus \overline{A}$ and $V \subset \partial D$ be an open neighborhood of A such that $z \notin V$. Take $(z'_m)_m \subset \partial D \setminus \overline{V}$ converging to z.

By Corollary 2.4, $\omega(., V, D) \in PSH(D)$, by Proposition 2.3, $\omega^*(., V, D) = -1$ on V. By Proposition 2.2 there is $z_m \in B(z'_m, 1/m) \cap D \setminus \hat{A}$ such that $\omega(z_m, V, D) > -2^{-m}$. Clearly

$$z_m \to z \qquad \text{when } m \to \infty$$

$$\omega(z_m, V, D) \to 0 \qquad \text{when } m \to \infty.$$

For $j > 0$ we set $u_j = 2^j \omega(., V, D)$. As $u_j(z_m)$ tends to zero then there is $M > 1$ such that $u_j(z_m) > -1$ for all $m > M$. As the points $z_1, \cdots, z_M$ do not belong to $\hat{A}$ then there is $u \in PSH(D)^-$, $u = -\infty$ on A such that $u(z_m) > -1$ for $0 < m \leq M$. We set

$$v_j = 2^{-j} \max\{u_j, u\}.$$

Remark that $v_j(z_m) > -2^{-j}$ for all $m > 0$ and $v_j^* = -1$ on A. Set

$$v(y) = \sum_{j=1}^{\infty} v_j(y) \qquad (y \in D).$$

Note that $-1 \leq -\sum 2^{-j} \leq v(z_m)$ for all $m > 0$, v is negative in D and $v^*|A \equiv -\infty$. v is the limit of a decreasing sequence of plurisubharmonic functions (its partial sums). Since v is not identically $-\infty$, we conclude that $v \in PSH(D)$ and $-1 \leq v^*(z)$ this means that $z \notin \hat{A}$. □

Remark 3.9. If A is an F_σ set, then $\hat{A} \cap \partial D = A$.

Proposition 3.10. *Let $D \subset \mathbb{C}^n$ be a B-regular domain and let $A \subset \partial D$ be b-pluripolar. Then*

$$\hat{A} \cap \partial D = A.$$

Proof. Obviously $A \subset \hat{A} \cap \partial D$. Now let $z \in \partial D \setminus A$. As A is b-pluripolar there exists $u \in \mathrm{PSH}(D)$ such that $u < -1$, $u^* = -\infty$ on A. If $u^*(z)$ is finite, there is nothing to prove. We will assume $u^*(z) = -\infty$ and construct a function $v \in \mathrm{PSH}(D) \cap C(\overline{D} \setminus \{z\})$ so that $u + v < 0$ in D and $(u+v)^*(z)$ is finite. This then shows that $z \notin \hat{A}$. Let

$$E_z(j) = \{w \in \partial D : \ \frac{1}{4j+1} \leq |z - w| \leq \frac{1}{4j}\}.$$

Because u^* is usc on ∂D and A is b-pluripolar, on $E_z(j)$, u^* assumes a maximum $M_j \leq -1$ at $w_j \in E_z(j)$ since $E_j(z)$ is not b-pluripolar then $M_j > -\infty$. Let $f_j \leq 0$ be continuous on ∂D, $f_j > u^*$ and $f_j(w_j) < u^*(w_j) + 1$ and let $0 \leq \chi_j \leq 1$ be a continuous function on ∂D with $\chi_j(w_j) = 1$ and compactly supported in $\frac{1}{4j+2} \leq |z - w| \leq \frac{1}{4j-1}$, and of sufficiently small size to be determined later. Then

$$u^* \leq \sum_{j=1}^{\infty} f_j \chi_j \quad \text{on } \partial D. \tag{1}$$

and

$$u^*(w_j) \geq \sum_{j=1}^{\infty} f_j \chi_j(w_j) - 1 \quad \text{for every } j. \tag{2}$$

Let F_j be the harmonic function on D, continuous on ∂D with boundary values $-f_j\chi_j$. The series $\sum_{j=1}^{\infty} F_j$ represents a monotonically increasing sequence of harmonic functions that are continuous up to ∂D. By choosing the support of χ_j sufficiently small, we can achieve, in view of Harnack's theorem, that the series converges uniformly on compact sets in $\overline{D} \setminus \{z\}$ and represents a harmonic function on D that is continuous on $\overline{D} \setminus \{z\}$ and has boundary values $\sum_{j=1}^{\infty} -f_j\chi_j$.

Now let $v_j = u_{F_j}$ be the Perron-Bremermann function of $-f_j\chi_j$. Then $0 \leq v_j = v_j^* \leq F_j$ on $\overline{D}$ with equality on ∂D because D is B-regular, and v_j is a continuous plurisubharmonic function. It follows that the series $v = \sum_{j=1}^{\infty} v_j$ is also uniformly convergent on compact sets in $\overline{D} \setminus \{z\}$, hence it represents a plurisubharmonic function that is continuous up to $\partial D \setminus \{z\}$ with boundary values $\sum_{j=1}^{\infty} F_j$ on $\partial D \setminus \{z\}$. Then by (1) and (2) we have

$$u^* + v = \lim_{k \to \infty} (u^* + \sum_{j=1}^{k} v_k) \leq 0 \quad \text{and} \quad u^*(w_j) + v(w_j) \geq -1 \text{ for all } j.$$

Because $u^* + v^*$ is usc, we have that $u^*(z) + v^*(z) \geq -1$. □

Remark 3.11. Of course, if the domain is not B-regular, Proposition 3.10 is no longer valid. Set $\Delta = \{0 < |z| < 1, |w| < 1\}$ and let $A = \{(z,1), |z| = 1\}$, then A is b-pluripolar, which is seen by considering $\log|w-1|$ and $\hat{A} = \{(z,1), |z| \leq 1\}$. The same applies for domains with (fine) analytic discs in the boundary, cf. [10].

Theorem 3.12. *Let $D \subset \mathbb{C}^n$ be B-regular and $A \subset \partial D$ be a b-pluripolar set. Then*

$$\hat{A} = A \cup \{z \in D, \quad \omega(z, A, D) < 0\}.$$

Proof. Combine Proposition 3.5 and Proposition 3.10 above. □

Conjecture 3.13. Let $A \subset \partial D$ whose compact sets are all b-pluripolar, then A must be b-pluripolar.

4. Completeness of b-pluripolar sets

Definition 4.1. We say that a subset $A \in \partial D$ is *completely b-pluripolar* if there exists a $u \in \mathrm{PSH}(D)$, $u < 0$, $\quad u \not\equiv -\infty$, such that $\{z \in \partial D, \quad u^*(z) = -\infty\} = A$.

Zeriahi,[14] gave conditions under which a pluripolar set is completely pluripolar. Here we adapt Zeriahi's result to boundary pluripolar sets. Our result requires only minor adaptations.

Proposition 4.2. *Let $D \subset \mathbb{C}^n$ be a B-regular domain and $A \subset \partial D$ be a b-pluripolar set. Suppose that F and K are compact subsets of $\overline{D}$ with $F \subset \hat{A}$ and $K \subset \overline{D} \setminus \hat{A}$. Then for all $C > 0$ there exists $\psi_K \in \mathrm{PSH}(D) \cap C(\overline{D})$ so that $\psi_K < 0$, $\psi_K < -C$ on F, and $\psi_K > -1$ on K.*

Proof. Let $a \in K \subset \overline{D} \setminus \hat{A}$. Then there exists $u \in \mathrm{PSH}(D)$ and negative so that $u^* = -\infty$ on $\hat{A}$ and $u^*(a) > -\infty$. Set $M = \sup\{u^*(z) - u^*(a), z \in \overline{D}\}$. Then

$$w(z) = \frac{u(z) - u^*(a)}{2(|M| + 1)} - 1/2, \qquad \text{for } z \in D,$$

is plurisubharmonic and $w < 0$ on D, $w^*|\hat{A} = -\infty$, $w^*(a) = -1/2$. By [13], Theorem 4.1, we can find a sequence in $\mathrm{PSH}(D) \cap C(\overline{D})$ that decreases to w^* on $\overline{D}$. In particular there exists in view of Dini's theorem an $f_a \in \mathrm{PSH}(D) \cap C(\overline{D})$ and negative such that $f_a < -C$ on F and $f_a(a) \geq w^*(a) = -1/2 > -1$. Then there exists a neighborhood V_a of a so that $f_a(z) > -1$ for all $z \in V_a$. By compactness we can find a finite subset of $I \subset K$ such that $K \subset \cup_{a \in I} V_a$. Set $\psi_K = \max\{f_a, a \in I\}$ then

$$\psi_K < 0, \quad \psi_K \in \mathrm{PSH}(D) \cap C(\overline{D}), \quad \psi_K(z) < -C \text{ for all } z \in F, \quad \text{and } \psi_K > -1 \text{ on } K.$$

□

Lemma 4.3. *Let D be a B-regular domain in $\mathbb{C}^n$. Let $A \subset \partial D$ be b-pluripolar and let $K \subset \partial D \setminus A$ be compact. Then there exists a $L \subset D \setminus \hat{A}$ such that every element of K is limit of a sequence in L and $L \cup K$ is compact.*

Proof. As K is compact there exist for every $j \in \mathbb{N}$ N_j points $z_{jl} \in K$, $1 \leq l \leq N_j$ such that $K \subset \cup_{l=1}^{N_j} B(z_{jl}, 1/j)$. Because $\hat{A}$ has empty interior, we can find a point $w_{jl} \in D \cap B(z_{jl}, 1/j) \setminus \hat{A}$. Now let $L = \{w_{lj} : 1 \leq l \leq N_j, j \in \mathbb{N}\}$. Then the limit points of L belong to K hence $K \cup L$ and if $z \in K \cap B(z_{lj}, 1/j)$ then $|z - w_{lj}| < 2/j$, therefore z is a limit of a subsequence of L. □

Theorem 4.4. *Let D be a B-regular domain in $\mathbb{C}^n$. Let $A \subset \partial D$ be b-pluripolar, F an F_σ set, G a G_δ set in ∂D such that $F \subset A \subset G$. Then there exists an $E \subset \partial D$ and a negative function $\psi \in \mathrm{PSH}(D)$ such that $F \subset E \subset G$, where $E = \{z \in \partial D : \ \psi^*(z) = -\infty\}$.*

Proof. Set $F = \cup_j F_j$ where $(F_j)_{j \geq 1}$ is an increasing sequence of compact sets in $\hat{A}$, and $\partial D \setminus G = \cup_j \tilde{K}_j$ where $(\tilde{K}_j)_j$ is an increasing sequence of compact sets in $\partial D \setminus G$. By Lemma 4.3 each $\tilde{K}_j$ can be enlarged to a compact set $K_j \subset \overline{D} \setminus \hat{A}$. Replacing K_{j+1} by $K_{j+1} \cup K_j$ if necessary, we can assume $K_j \subset K_{j+1}$. By Proposition 4.2 for each $j > 0$ there exists $\psi_j \in \mathrm{PSH}(D) \cap C(\overline{D})$ with

(3) $$\psi_j \leq -2^j \text{ on } F_j, \qquad \text{and} \qquad \psi_j \geq -1 \text{ on } K_j.$$

The function $\psi = \sum_{j=1}^{\infty} 2^{-j}\psi_j$ is negative. For $z \in \partial D \setminus G$ there is $J > 0$ so that $z \in K_J$ and a sequence $(z_m)_m \subset K_J$ converging to z we find that for all m

$$\psi(z_m) = \sum_{j=1}^{J} 2^{-j}\psi_j(z_m) + \sum_{j=1+J}^{\infty} 2^{-j}\psi_j(z_m) \geq \inf_{K_J} \sum_{j=1}^{J} 2^{-j}\psi_j - 1 > -C_J > -\infty, \tag{4}$$

where C_J depends only on K_J, in view of the continuity of the ψ_j. It follows that ψ is plurisubharmonic on D as limit of a decreasing sequence of plurisubharmonic functions. It satisfies $\psi^* \equiv -\infty$ on F because of (3). Finally if $z \in \partial D \setminus G$, then $z \in \overline{K_j \cap D}$ for some j and by (4) $\psi^*(z) > C_j$, hence $\psi^* > -\infty$ on $\partial D \setminus G$. Set $E = \{z \in \partial D, \psi^*(z) = -\infty\}$ then $F \subset E \subset G$. □

Corollary 4.5. *Let D be a B-regular domain in $\mathbb{C}^n$. Every b-pluripolar set $A \subset \partial D$ that is a G_δ as well as an F_σ is completely b-pluripolar.*

Proof. By Proposition 3.10 $\hat{A} \cap \partial D = A$. We apply Theorem 4.4 with $F = A = G$. The theorem gives us a negative $\psi \in \mathrm{PSH}(D)$ with $A = \{z \in \partial D \text{ with } \psi^*(z) = -\infty\}$. In particular, $\psi \not\equiv -\infty$ on D and A is completely b-pluripolar. □

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