

## COUNTERPART OF THE AUBIN–YAU FUNCTIONAL FOR REAL RIEMANNIAN MANIFOLDS

BY DONGWEI GU

**Abstract.** In this note we construct a functional which is an analogue of the Aubin–Yau functional on any connected compact real Riemannian manifold. The construction is similar to the one in the Kähler case and they coincide with each other when the manifold is of complex dimension 1.

**1. Introduction.** Let  $(X, \omega)$  be a connected compact Kähler manifold of complex dimension  $n$  with the Kähler form  $\omega$ . We define  $\mathcal{H}$  to be the space of Kähler potentials:

$$\mathcal{H} = \{\varphi : X \rightarrow \mathbb{R}, \varphi \in C^\infty(X) \mid \omega_\varphi := \omega + i\partial\bar{\partial}\varphi > 0\}.$$

It is of interest to explore the geometric structure of this space  $\mathcal{H}$ . Firstly, Mabuchi [7] discovered a natural infinite-dimensional Riemannian manifold structure on  $\mathcal{H}$ . And then Donaldson [4] showed that  $\mathcal{H}$  is actually a non-positively curved symmetric space of non-compact type.

The “ $\partial\bar{\partial}$ -lemma” says that any other Kähler form cohomologous to  $\omega$  can be expressed by a Kähler potential. We define  $\mathcal{H}_0$  to be the space of Kähler forms:

$$\mathcal{H}_0 = \{\omega_\varphi = \omega + i\partial\bar{\partial}\varphi \mid \varphi \in \mathcal{H}\}.$$

It is easy to see that  $\mathcal{H}_0 = \mathcal{H}/\mathbb{R}$ . In fact, there is also a Riemannian decomposition:

$$(1) \quad \mathcal{H} = \mathcal{H}_0 \times \mathbb{R}.$$

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2010 *Mathematics Subject Classification.* 58B20, 58E11.

*Key words and phrases.* Infinite-dimensional manifold, volume element, Aubin–Yau functional.

The author is supported by international PhD programme “Geometry and Topology in Physical Models” of the Foundation for Polish Science.

The way to give rise to this Riemannian decomposition (1) is to make a right normalization on  $\mathcal{H}$ , which is done by the Aubin–Yau functional (see Donaldson [4], Błocki [2]). This useful functional was first introduced by Aubin [1].

Let us recall some background on energy functionals occurring in Kähler geometry. The Mabuchi  $\mathcal{K}$ -energy functional, established by Mabuchi [6], plays an important role in the study of metrics of constant scalar curvatures. This functional provides a variational approach to solving the equations associated to metrics of constant scalar curvatures rather than dealing with a fourth order nonlinear partial differential equation. There is another important functional in this theory, the so called Aubin–Yau functional. In [9], Tian showed that this functional is closely related to an analytic criterion for the existence of Kähler–Einstein metrics on Kähler manifolds with positive first Chern class. Remarkably, it turns out that this functional is also closely related to all forms of GIT stability conditions (see Phong and Sturm [8] for a survey).

Now let us discuss the case of real Riemannian manifolds. Let  $(M, g)$  be a connected compact Riemannian manifold of real dimension  $m$  with the Riemannian metric  $g$ . Donaldson [5] introduced an interesting program in the space of “volume potentials” which we will define later, similar to the program in Kähler case. It seems natural since when  $M$  is 2-dimensional, it can be seen as a complex manifold of complex dimension 1, and then these two programs coincide exactly. He also shown that this space admits an infinite dimensional Riemannian manifold structure with non-positive sectional curvature. He has asked whether there is a smooth geodesic between any two points in this space. In fact, the existence of such geodesic segment is related to some other problems in partial differential equations such as Nahm’s equation, regularity for some free boundary problems. So it is also worthwhile to study the geometric structure of this space.

**2. Setting for the real case.** We follow the notations in Section 1. By a volume element on  $M$  we mean a differential form of degree  $m$ , positive everywhere. In local coordinates  $(x_1, x_2, \dots, x_m)$ , such a volume element  $\sigma$  takes the form:

$$\sigma = f(x)dx, \text{ where } dx = dx_1 \wedge dx_2 \wedge \dots \wedge dx_m \text{ and } f(x) > 0.$$

In local coordinates, we can write the Riemannian metric  $g$  as follows:

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

There is a canonical volume element  $dg$  coming from this metric  $g$ :

$$dg := \sqrt{\det(g_{ij})} dx_1 \wedge dx_2 \wedge \dots \wedge dx_m,$$

where  $\det(g_{ij})$  refers to the determinant of the  $m \times m$  matrix  $(g_{ij})$ . Now we define a space  $\mathcal{V}_0$ , that is the space of volume elements on  $(M, g)$  with fixed total volume  $\text{Vol} := \int_M dg$ . Since  $M$  is closed, this space can be expressed as:

$$\mathcal{V}_0 := \{(1 + \Delta\phi)dg \mid \phi \in C^\infty(M), 1 + \Delta\phi > 0\},$$

where  $\Delta$  is the Laplacian operator with respect to  $g$ . By this expression, every element in  $\mathcal{V}_0$  is determined by some “potential” function up to a constant. So we can define a space  $\mathcal{V}$  of such “potential” functions for the volume elements:

$$\mathcal{V} := \{\phi \in C^\infty(M) \mid 1 + \Delta\phi > 0\}.$$

Clearly,  $\mathcal{V}_0 = \mathcal{V}/\mathbb{R}$ . In fact,  $\mathcal{V}$  admits a Riemannian decomposition too, which we will explain in Section 3.

Following [5], now we explain how to make  $\mathcal{V}$  into a Riemannian manifold. At each point  $\phi_0$  of  $\mathcal{V}$ , we can identify  $C^\infty(M)$  with the tangent space  $T_{\phi_0}\mathcal{V}$  of  $\mathcal{V}$  at  $\phi_0$  via the following isomorphism:

$$\begin{aligned} C^\infty(M) &\cong T_{\phi_0}\mathcal{V} \\ \psi &\leftrightarrow \left. \frac{d}{ds} \right|_{s=0} (\phi_0 + s\psi), \end{aligned}$$

where  $s \in [-\epsilon, \epsilon] \mapsto \phi_0 + s\psi \in \mathcal{V}$  is a smooth path in  $\mathcal{V}$  with a sufficiently small  $\epsilon > 0$ . On  $T_{\phi_0}\mathcal{V}$ , we can define a scalar product:

$$\langle\langle \psi, \eta \rangle\rangle_{\phi_0} := \frac{1}{\text{Vol}} \int_M \psi\eta(1 + \Delta\phi_0)dg, \quad \psi, \eta \in T_{\phi_0}\mathcal{V}.$$

In terms of this, the norm of a tangent vector  $\psi \in T_{\phi_0}\mathcal{V}$  is given by:

$$\|\psi\|_{\phi_0}^2 := \langle\langle \psi, \psi \rangle\rangle_{\phi_0} = \frac{1}{\text{Vol}} \int_M \psi^2(1 + \Delta\phi_0)dg.$$

For a smooth path  $\phi(t) : [0, 1] \rightarrow \mathcal{V}$ , which is simply a smooth function on  $M \times [0, 1]$ , the “energy” of this path is:

$$(2) \quad E(\phi(t)) := \frac{1}{2} \int_0^1 \|\dot{\phi}\|_{\phi(t)}^2 dt = \frac{1}{2\text{Vol}} \int_0^1 \int_M |\dot{\phi}|^2(1 + \Delta\phi)dgdt,$$

where we denote  $\dot{\phi} = \frac{d\phi}{dt}$ ,  $\ddot{\phi} = \frac{d^2\phi}{dt^2}$  and so on. To find the geodesic equation in  $\mathcal{V}$ , we consider the Euler–Lagrange equation associated to energy functional (2). For readers’ convenience, we give the computation here. Let  $\psi(t)$  be a

small variation, then

$$\begin{aligned}
& \left. \frac{d}{ds} \right|_{s=0} E(\phi(t) + s\psi(t)) \\
&= \frac{1}{2 \text{Vol}} \int_0^1 \int_M (2\dot{\phi}\dot{\psi}(1 + \Delta\phi) + |\dot{\phi}|^2 \Delta\psi) dg dt \\
&= \frac{1}{2 \text{Vol}} \int_0^1 \int_M (\psi(-2\frac{d}{dt}(\dot{\phi}(1 + \Delta\phi)) + \Delta(|\dot{\phi}|^2)\psi) dg dt \\
&= \frac{1}{2 \text{Vol}} \int_0^1 \int_M (\psi(-2\ddot{\phi}(1 + \Delta\phi) - 2\dot{\phi}\Delta\dot{\phi} + (2\dot{\phi}\Delta\dot{\phi} + 2|\nabla\dot{\phi}|^2)) dg dt \\
&= \frac{1}{2 \text{Vol}} \int_0^1 \int_M \psi(-2\ddot{\phi}(1 + \Delta\phi) + 2|\nabla\dot{\phi}|^2) dg dt
\end{aligned}$$

where we use integration by parts in the second equality and  $|\nabla\dot{\phi}|$  means the norm of gradient of function  $\dot{\phi}$  with respect to the metric  $g$ . Then the geodesic equation is:

$$(3) \quad \ddot{\phi}(1 + \Delta\phi) - |\nabla\dot{\phi}|^2 = 0.$$

This equation was investigated in [3], where the authors considered a perturbed equation (3), that is for any  $\epsilon > 0$ ,

$$(4) \quad \ddot{\phi}(1 + \Delta\phi) - |\nabla\dot{\phi}|^2 = \epsilon.$$

They have proved that for any two points  $\phi_0, \phi_1 \in \mathcal{V}$ , there exists a smooth solution of (4),  $\Phi(t) : [0, 1] \rightarrow \mathcal{V}$  which connects  $\phi_0$  and  $\phi_1$ . In particular, they established the *weak*  $C^2$  estimate (that is,  $\Delta\Phi$ ,  $\Phi_{tt}$ , and  $\nabla\Phi_t$  are bounded, while  $\nabla^2\Phi$  does not have to) of the solution independent of  $\inf \epsilon$ . As a result, they showed that for any two points  $\phi_0, \phi_1 \in \mathcal{V}$ , there exists a weakly  $C^2$  geodesic  $\Phi(t) : [0, 1] \rightarrow \bar{\mathcal{V}}$ , which connects  $\phi_0$  and  $\phi_1$ , where  $\bar{\mathcal{V}}$  means the closure of  $\mathcal{V}$  under the weak  $C^2$  topology.

Geodesic equation (3) shows us exactly how to define the ‘‘Levi-Civita’’ connection on  $T\mathcal{V}$ . Let  $\phi(t)$  be a smooth path in  $\mathcal{V}$  and  $\psi(t)$  be another smooth function on  $M \times [0, 1]$ , which we regard as a vector field along the path  $\phi(t)$ . Then the covariant derivative of  $\psi$  along the path  $\phi(t)$  is given by:

$$(5) \quad \nabla_{\dot{\phi}}\psi := \dot{\psi} - \frac{1}{1 + \Delta\phi} \langle \nabla\psi, \nabla\dot{\phi} \rangle,$$

where  $\langle \cdot, \cdot \rangle$  means the inner product on  $TM$  with respect to  $g$ . This connection is torsion free because in the local ‘‘coordinate chart’’, which represent  $\mathcal{V}$  as an

open subset of  $C^\infty(M)$ , the ‘‘Christoffel symbol’’ at  $\phi \in \mathcal{V}$  is just:

$$\begin{aligned}\Gamma : C^\infty(M) \times C^\infty(M) &\rightarrow C^\infty(M) \\ \Gamma(\psi_1, \psi_2) &= \frac{-1}{1 + \Delta\phi} \langle \nabla\psi_1, \nabla\psi_2 \rangle\end{aligned}$$

which is symmetric in  $\psi_1, \psi_2$ . This connection is also compatible with the metric, since:

$$\begin{aligned}\frac{d}{dt} \langle \langle \psi_1, \psi_2 \rangle \rangle_\phi &= \frac{d}{dt} \left( \frac{1}{\text{Vol}} \int_M \psi_1 \psi_2 (1 + \Delta\phi) dg \right) \\ &= \frac{1}{\text{Vol}} \int_M \left( (\dot{\psi}_1 \psi_2 + \psi_1 \dot{\psi}_2) (1 + \Delta\phi) + \psi_1 \psi_2 \Delta\dot{\phi} \right) dg \\ &= \frac{1}{\text{Vol}} \int_M \left( (\dot{\psi}_1 \psi_2 + \psi_1 \dot{\psi}_2) (1 + \Delta\phi) - \langle \nabla\psi_1, \nabla\dot{\phi} \rangle \psi_2 - \langle \nabla\psi_2, \nabla\dot{\phi} \rangle \psi_1 \right) dg \\ &= \frac{1}{\text{Vol}} \int_M \left( \left( \dot{\psi}_1 - \frac{1}{1 + \Delta\phi} \langle \nabla\psi_1, \nabla\dot{\phi} \rangle \right) \psi_2 \right. \\ &\quad \left. + \left( \dot{\psi}_2 - \frac{1}{1 + \Delta\phi} \langle \nabla\psi_2, \nabla\dot{\phi} \rangle \right) \psi_1 \right) (1 + \Delta\phi) dg \\ &= \langle \langle \nabla_{\dot{\phi}} \psi_1, \psi_2 \rangle \rangle_\phi + \langle \langle \psi_1, \nabla_{\dot{\phi}} \psi_2 \rangle \rangle_\phi.\end{aligned}$$

We will use these properties of this ‘‘Levi–Civita’’ connection in the next Section.

**3. Normalization and Construction of the functional.** Now we want to induce a Riemannian structure on  $\mathcal{V}_0$  from the structure on  $\mathcal{V}$ . Since  $\mathcal{V}_0 = \mathcal{V}/\mathbb{R}$ , we just need a proper normalization on  $\mathcal{V}$ . It turns out that we can do it by constructing a functional which is the counterpart of the Aubin–Yau functional in Kähler case. Specifically, we look for a functional  $I$  which is characterized by the following properties:

$$(6) \quad I(0) = 0, \quad dI_\phi(\psi) = \frac{1}{\text{Vol}} \int_M \psi (1 + \Delta\phi) dg, \quad \phi \in \mathcal{V}, \quad \psi \in C^\infty(M).$$

We can regard  $dI$  as a 1-form on  $\mathcal{V}$ . The point is that such a functional always exists if this 1-form is closed. Indeed:

$$\begin{aligned}(d(dI_\phi))_\phi(\psi_1, \psi_2) &= d(dI_\phi(\psi_1))_\phi(\psi_2) - d((dI_\phi(\psi_2))_\phi)(\psi_1) \\ &= \frac{1}{\text{Vol}} \int_X (\psi_1 \Delta\psi_2 - \psi_2 \Delta\psi_1) dg = 0.\end{aligned}$$

It follows that there is a functional  $I$  satisfying (6). For any smooth path  $\phi(t)$  in  $\mathcal{V}$  joining 0 with  $\phi$ , we can write  $I$  formally:

$$I(\phi) = \int_0^1 \frac{1}{\text{Vol}} \int_M \dot{\phi}(1 + \Delta\phi) dg dt.$$

The fact  $dI$  is closed implies that  $I(\phi)$  is independent of the choice of the path  $\phi(t)$ . Taking  $\phi(t) = t\phi$ , we obtain the formula explicitly:

$$(7) \quad I(\phi) = \frac{1}{\text{Vol}} \int_M \left( \phi + \frac{1}{2} \phi \Delta\phi \right) dg,$$

which coincides with the Kähler case when the real dimension of  $M$  is 2. We can easily get

$$I(\phi + c) = I(\phi) + c$$

for any real constant  $c$ .

For any smooth path  $\phi(t)$  in  $\mathcal{V}$  joining 0 with  $\phi$ , by (5) and the last formula in Section 2, we have

$$\frac{d^2}{dt^2} I(\phi) = \frac{d}{dt} \langle \dot{\phi}, 1 \rangle_{\phi(t)} = \langle \nabla_{\dot{\phi}} \dot{\phi}, 1 \rangle_{\phi(t)}$$

which yields that  $I$  is affine along geodesics. And  $\phi$  is a geodesic implies that also  $\phi - I(\phi)$  is a geodesic. We call a “volume potential”  $\phi$  *normalized* if  $I(\phi) = 0$ . Then any volume element in  $\mathcal{V}_0$  has a unique normalized “volume potential” in  $\mathcal{V}$ , and the restriction of the metric on  $\mathcal{V}$  to  $I^{-1}(0)$  endows  $\mathcal{V}_0$  with a Riemannian structure which is independent of the choice of  $dg$  as long as the total volume of  $M$  is fixed. And clearly the tangent space of  $\mathcal{V}_0$  at a point  $(1 + \Delta\phi)dg$  written as  $dg_\phi$  can be realized as:

$$T_{dg_\phi} \mathcal{V}_0 = \left\{ \psi \in C^\infty(M) \mid \int_M \psi dg_\phi = 0 \right\}.$$

To summarize, we now state the result we have just obtained:

**THEOREM.** *Let  $(M, g)$  be a connected compact real Riemannian manifold and  $\mathcal{V}$  be the space of functions on  $M$  such that  $\mathcal{V} = \{\phi \in C^\infty(M) \mid 1 + \Delta\phi > 0\}$ , which admits an infinite-dimensional Riemannian manifold structure. Then there is a functional  $I$  given by*

$$I(\phi) = \frac{1}{\text{Vol}} \int_M \left( \phi + \frac{1}{2} \phi \Delta\phi \right) dg, \quad \phi \in \mathcal{V}$$

*such that  $I^{-1}(0)$  is a totally geodesic subspace of  $\mathcal{V}$ . Moreover, if we let  $\mathcal{V}_0$  be the space of volume elements on  $M$  with fixed total volume  $\int_M dg$ , then the bijective mapping*

$$I^{-1}(0) \ni \phi \mapsto (1 + \Delta\phi)dg \in \mathcal{V}_0$$

induces a Riemannian structure on  $\mathcal{V}_0$  with tangent space of  $\mathcal{V}_0$  at a point  $d\nu$  realized as

$$T_{d\nu}\mathcal{V}_0 = \left\{ \psi \in C^\infty(M) \mid \int_M \psi d\nu = 0 \right\}.$$

Thus there is a Riemannian decomposition  $\mathcal{V} = \mathcal{V}_0 \times \mathbb{R}$ .

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Received April 30, 2015

Faculty of Mathematics and Computer Science  
 Jagiellonian University  
 Łojasiewicza 6  
 30-348 Kraków  
 Poland  
 e-mail: dongwei.gu@im.uj.edu.pl