WEAK SOLUTIONS TO THE COMPLEX MONGE-AMPÈRE EQUATION ON HERMITIAN MANIFOLDS

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Dedicated to Duong H. Phong on the occasion of his 60th birthday

Abstract. The main result asserts the existence of continuous solutions of the complex Monge-Ampère equation with the right hand side in $L^p, p > 1$, on compact Hermitian manifolds.

Introduction

Let (X, ω) be a compact Hermitian manifold of complex dimension n. We study the weak solutions to the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = f\omega^n, \quad \omega + dd^c\varphi \geq 0,$$

where $0 \leq f \in L^p(X, \omega^n)$, $p > 1$, and $d^c = \frac{i}{2\pi}(\bar\partial - \partial)$, $dd^c = \frac{i}{\pi}\partial\bar\partial$, with the displayed inequality understood in the sense of currents.

We follow the pluripotential approach introduced by S. Dinew and the first author in [7], where L^∞ estimates for the above equation were obtained. Here we refine those estimates and prove the existence of continuous solutions.

Theorem 0.1. *Let (X, ω) be a compact Hermitian manifold, $dim\, X = n$. Let $0 \leq f \in L^p(X, \omega^n)$, $p > 1$, be such that $\int_X f\omega^n > 0$. There exist a constant $c > 0$ and a function $u \in C(X)$ satisfying the equation*

$$(\omega + dd^c u)^n = cf\,\omega^n, \quad \omega + dd^c u \geq 0,$$

in the weak sense.

The main tool is a generalized version of the comparison principle due to Bedford-Taylor [1, 2]. We call it *modified comparison principle* just for a convenient reference. In its formulation we use a constant $B > 0$ such that

$$\begin{cases} -B\omega^2 \leq 2n dd^c\omega \leq B\omega^2, \\ -B\omega^3 \leq 4n^2 d\omega \wedge d^c\omega \leq B\omega^3. \end{cases} \tag{0.1}$$

We denote by $PSH(\omega)$ the set of ω-plurisubharmonic functions on X (see Section 1).

Theorem 0.2 (modified comparison principle)**.** *Let (X, ω) be a compact Hermitian manifold and suppose that $\varphi, \psi \in PSH(\omega) \cap L^\infty(X)$. Fix $0 < \varepsilon < 1$ and set $m(\varepsilon) = \inf_X[\varphi - (1-\varepsilon)\psi]$. Then, for any $0 < s < \frac{\varepsilon^3}{16B}$, we have*

$$\int_{\{\varphi < (1-\varepsilon)\psi + m(\varepsilon) + s\}} \omega^n_{(1-\varepsilon)\psi} \leq (1 + \frac{s}{\varepsilon^n} C) \int_{\{\varphi < (1-\varepsilon)\psi + m(\varepsilon) + s\}} \omega^n_\varphi,$$

where C is a uniform constant depending only on n, B.

It was shown in [7] that the comparison principle which is valid on Kähler manifolds (see [17]) is no longer true on general Hermitian manifolds.

The complex Monge-Ampère equation on complex Hermitian manifolds was first studied by Cherrier [4, 5, 6] and Hanani [13, 14]. There has been a renewed interest recently in the works of Guan - Li [11] and Tosatti - Weinkove [21, 22]. The breakthrough was made by Tosatti and Weinkove [22] who proved the existence and uniqueness of the smooth solution to complex Monge-Ampère equation on a general compact Hermitian manifold. Since then more papers appeared (e.g., [9], [12], [19], [20], [27]), some in relation to the Chern-Ricci flow. It was shown in [9, 10], [23, 24, 25] that the flow enjoys many common properties with the Kähler-Ricci flow. In the study of the latter the weak solutions of the complex Monge-Ampère equation play an important role, and thus the investigation of the Hermitian case seems to be well motivated.

The method based on the modified comparison principle can also be applied in the case when $X = \Omega$ is a bounded open set in $\mathbb{C}^n$. We consider the Dirichlet problem for the Monge-Ampère operator and generalize the stability estimates [15] from the Kähler setting to the Hermitian one.

Corollary 0.3. *Consider Ω be a bounded open set in $\mathbb{C}^n$ and ω be a Hermitian metric in $\mathbb{C}^n$. Let $u, v \in PSH(\omega) \cap C(\bar{\Omega})$ be such that*

$$(\omega + dd^c u)^n = f\omega^n, \quad (\omega + dd^c v)^n = g\omega^n,$$

with $0 \leq f, g \in L^p(\Omega, \omega^n)$, $p > 1$. Then

$$\|u - v\|_{L^\infty(\bar{\Omega})} \leq \sup_{\partial\Omega} |u - v| + C\|f - g\|^{\frac{1}{n}}_{L^p(\omega^n)},$$

where C depends only on Ω, ω and p.

Thanks to the domination principle and the stability estimate the Dirichlet problem for Monge-Ampère operator (with the background metric ω) is solvable for the right hand side in L^p, $p > 1$.

Corollary 0.4. *There exists a unique continuous solution to the Dirichlet problem* (4.1) *in a C^∞ strictly pseudoconvex domain.*

The note is organized as follows. We recall some basic properties of ω- plurisubharmonic functions on complex Hermitian manifolds in *Section 1*. *Section 2* is devoted to prove the modified comparison principle. Then the domination principle in the local case is inferred in *Section 3*. The stability estimates and the Dirichlet problem for complex Monge-Ampère in a bounded domain in $\mathbb{C}^n$ are studied in *Section 4*. In *Section 5* we show L^∞ *a priori* estimates and the existence of continuous solutions to complex Monge-Ampère equations on a compact Hermitian manifold.

Dedication It is a great honour for the authors to dedicate this paper to Duong H. Phong in appreciation of his wisdom which reaches far beyond mathematics.

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1. Basic properties of ω-psh functions in the Hermitian setting

Let Ω be an open set in $\mathbb{C}^n$ and ω a Hermitian metric in $\mathbb{C}^n$. We collect here some basic facts about ω-plurisubharmonic (ω-psh for short) functions. We refer to [7] for more discussion. Recall that we use the normalisation $d = \partial + \bar{\partial}, d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$, $dd^c = \frac{i}{\pi}\partial\bar{\partial}$.

Definition 1.1. Let $u : \Omega \to [-\infty, +\infty[$ be a upper semi-continuous. Then u is called ω-psh if $u \in L^1_{loc}(\Omega, \omega^n)$ and $dd^c u + \omega \geq 0$ as a current.

Denote by $PSH(\Omega, \omega)$ the set of ω-psh functions in Ω (when Ω is clear from the context, we write $PSH(\omega)$). We often use the short-hand notation $\omega_u := (\omega + dd^c u)$. Following Bedford-Taylor [2], one defines the wedge product

$$\omega_{v_1} \wedge ... \wedge \omega_{v_k}$$

for $v_1, ..., v_k \in PSH(\omega) \cap L^\infty(\Omega)$, $1 \leq k \leq n$; proceeding by induction over k. For $k = 1$ the definition is given by classical distribution theory. Suppose that for $1 \leq k \leq n-1$ the current

$$T = \omega_{v_1} \wedge ... \wedge \omega_{v_k}$$

is well defined. Fix a small ball $\mathbb{B}$ in Ω and a strictly psh function ρ such that $dd^c\rho \geq 2\omega$ in $\mathbb{B}$. Put $\gamma = dd^c\rho - \omega$ and $u_l = \rho + v_l \in PSH(\mathbb{B}) \cap L^\infty(\mathbb{B})$, then T can be written in $\mathbb{B}$ as a linear combination of positive currents

$$dd^c u_{j_1} \wedge ... \wedge dd^c u_{j_l} \wedge \gamma^{k-l}, \quad 1 \leq j_1 < ... < j_l \leq k, \ 1 \leq l \leq k. \tag{1.1}$$

We know that there are sequences of smooth ω-psh function $\{v_l^j\}_{j=1}^\infty$ which decrease to v_l, $1 \leq l \leq k$ (by Demailly's regularization theorem for quasi-psh functions). Since T is a linear combination of positive currents of the form (1.1), we obtain by the results from [2]

$$T = \lim_{j\to\infty} T_j = \lim_{j\to\infty} \omega_{v_1^j} \wedge ... \wedge \omega_{v_k^j} \quad \text{weakly.}$$

Thus, T is a positive current of bidgree (k, k). Moreover,

$$dT = \sum_{l=1}^{k} d\omega \wedge \omega_{v_1} \wedge ...\widehat{\omega}_{v_l} ... \wedge \omega_{v_k};$$

$$d^cT = \sum_{l=1}^{k} d^c\omega \wedge \omega_{v_1} \wedge ...\widehat{\omega}_{v_l} ... \wedge \omega_{v_k};$$

$$dd^cT = 2\sum_{1\leq l<m\leq k} d\omega\wedge d^c\omega\wedge\omega_{v_1}\wedge...\widehat{\omega}_{v_l}...\widehat{\omega}_{v_m}...\wedge\omega_{v_k} + \sum_{l=1}^{k} dd^c\omega\wedge\omega_{v_1}\wedge...\widehat{\omega}_{v_l}...\wedge\omega_{v_n}.$$

The notation $\widehat{\omega}_{v_l}$ means that this term does not appear in the wedge product. Now we define for $u \in PSH(\omega) \cap L^\infty(\Omega)$

$$dd^c u \wedge T := dd^c(u \wedge T) - du \wedge d^cT + d^c u \wedge dT - udd^cT.$$

The right hand side is well defined by the above formulas for dT, d^cT and dd^cT. Let $\{u^j\}_{j=1}^\infty$ be a sequence of smooth ω-psh functions decreasing to u. We have

$$dd^c u \wedge T = \lim_{j\to\infty} dd^c u^j \wedge T_j \quad \text{weakly.}$$

Note here that for any test form φ of bidgree $(n-k-1, n-k-1)$

$$du \wedge d^cT \wedge \varphi = -d^c u \wedge dT \wedge \varphi.$$

Thus,

$$\omega_u \wedge T = \omega \wedge T + dd^c u \wedge T := \omega \wedge T + dd^c(uT) - 2du \wedge d^c T - u dd^c T$$

is a positive current of bidgree $(k+1, k+1)$. In the special case when $v_1 = ... = v_n = v \in PSH(\omega) \cap L^\infty(\Omega)$ we get the definition of Monge-Ampère operator

$$\omega_v^n := \omega_v \wedge ... \wedge \omega_v,$$

(n factors on the right hand side) which is a Radon measure. Finally, we state for the later reference a convergence result which follows also from the corresponding statement in [2] applied to currents of the form (1.1).

Proposition 1.2. *Let $v_1, ..., v_k \in PSH(\omega) \cap L^\infty(\Omega)$, $1 \leq k \leq n$. Suppose that the sequences of bounded ω-psh functions $\{v_1^j\}_{j=1}^\infty$, ...,$\{v_k^j\}_{j=1}^\infty$ decrease (or uniformly converge) to $v_1, ..., v_k$ respectively. Then*

$$\lim_{j\to\infty} \omega_{v_1^j} \wedge ... \wedge \omega_{v_k^j} = \omega_{v_1} \wedge ... \wedge \omega_{v_k} \quad \textit{weakly.}$$

In particular, if $\{u_j\}_{j=1}^\infty \in PSH(\omega) \cap L^\infty(\Omega)$ decreases (or uniformly converges) to $u \in PSH(\omega) \cap L^\infty(\Omega)$, then

$$\lim_{j\to\infty} \omega_{u_j}^n = \omega_u^n \quad \textit{weakly.}$$

Let now (X, ω) be a compact Hermitian manifold, with $dim_{\mathbb{C}} X = n$. The above (local) construction applies in this setting.

Definition 1.3. Let $u : X \to [-\infty, +\infty[$ be an upper semi-continuous function. Then, u is called ω-psh if $u \in L^1(X, \omega^n)$ and $dd^c u + \omega \geq 0$ as a current.

Denote by $PSH(\omega)$ the set of ω-psh functions on X. By the definition $u \in PSH(\omega)$ if and only if $u \in PSH(\Omega, \omega)$ for any coordinate chart $\Omega \subset\subset X$. Using partition of unity, we define the Monge-Ampère operators ω_u^n for $u \in PSH(\omega) \cap L^\infty(X)$. It is also clear that Proposition 1.2 holds in this setting.

2. The modified comparison principle

Let (X, ω) be a compact Hermitian manifold, $dim_{\mathbb{C}} X = n$. It is known (see [7]) that the comparison principle is not true on a general compact Hermitian manifold. We shall use two lemmata to prove the main theorem of this section (Theorem 2.3). From the proof of Proposition 3.1 in [1] and the approximation result in [3] we have the following statement.

Lemma 2.1. *For $T := (\omega + dd^c v_1) \wedge ... \wedge (\omega + dd^c v_{n-1})$, where $v_1, ..., v_{n-1} \in PSH(\omega) \cap L^\infty(X)$ and for $\varphi, \psi \in PSH(\omega) \cap L^\infty(X)$ we have*

$$\int_{\{\varphi<\psi\}} dd^c\psi \wedge T \leq \int_{\{\varphi<\psi\}} dd^c\varphi \wedge T + \int_{\{\varphi<\psi\}} (\psi - \varphi)\, dd^c T.$$

A weaker version of the comparison principle was shown in [7].

Lemma 2.2. *Let $\varphi, \psi \in PSH(\omega) \cap L^\infty(X)$. Then there is a constant $C_n = C(n)$ such that, for $B \sup_{\{\varphi<\psi\}}(\psi - \varphi) \leq 1$,*

$$\int_{\{\varphi<\psi\}} (\omega + dd^c\psi)^n \leq \int_{\{\varphi<\psi\}} (\omega + dd^c\varphi)^n + C_n B \sup_{\{\varphi<\psi\}} (\psi - \varphi) \sum_{k=0}^{n-1} \int_{\{\varphi<\psi\}} \omega_\varphi^k \wedge \omega^{n-k}.$$

We are ready to prove the modified comparison principle.

Theorem 2.3. *Let $\varphi, \psi \in PSH(\omega) \cap L^\infty(X)$. Fix $0 < \varepsilon < 1$ and set $m(\varepsilon) = \inf_X[\varphi - (1-\varepsilon)\psi]$. Then for any $0 < s < \frac{\varepsilon^3}{16B}$,*

$$\int_{\{\varphi<(1-\varepsilon)\psi+m(\varepsilon)+s\}} \omega^n_{(1-\varepsilon)\psi} \leq (1+\frac{sB}{\varepsilon^n}\, C) \int_{\{\varphi<(1-\varepsilon)\psi+m(\varepsilon)+s\}} \omega^n_\varphi,$$

where C is a uniform constant depending only on n.

Proof. We wish to apply Lemma 2.2 with $(1-\varepsilon)\psi + m(\varepsilon) + s$ in place of ψ. Note that on $U(\varepsilon, s) = \{\varphi < (1-\varepsilon)\psi + m(\varepsilon) + s\}$,

$$\sup_{U(\varepsilon,s)} [(1-\varepsilon)\psi + m(\varepsilon) - \varphi + s] \leq s.$$

Therefore, in view of Lemma 2.2, it is enough to estimate

$$\sum_{k=0}^{n-1} \int_{U(\varepsilon,s)} \omega^k_\varphi \wedge \omega^{n-k}.$$

For $k = 0, ..., n$, set

$$a_k = \int_{U(\varepsilon,s)} \omega^k_\varphi \wedge \omega^{n-k}.$$

Let $\delta := \frac{\varepsilon^3}{16B}$. We shall verify that for $0 < s < \delta$

$$\varepsilon\, a_0 \leq a_1 + \delta\, B\, a_0, \quad \text{and} \quad \varepsilon\, a_1 \leq a_2 + \delta\, B\,(a_1 + a_0), \tag{2.1}$$

and for $2 \leq k \leq n-1$,

$$\varepsilon\, a_k \leq a_{k+1} + \delta\, B\,(a_k + a_{k-1} + a_{k-2}). \tag{2.2}$$

Let us assume for a moment that (2.1) and (2.2) are true. It follows from the first inequality of (2.1) that

$$a_0 \leq d_1\, a_1 \ \text{ with } d_1 = \frac{1}{\varepsilon - \delta\, B}. \tag{2.3}$$

From the second inequality of (2.1) and (2.3) we have

$$a_0 \leq d_1\, d_2\, a_2 \quad \text{and} \quad a_1 \leq d_2\, a_2,$$

with $1/d_2 := \varepsilon - \delta\, B\,(1 + d_1)$. Using (2.2) and the induction we get that, for $k = 0, ..., n-1$,

$$a_k \leq d_{k+1} \ldots d_n\, a_n \tag{2.4}$$

where $d_0 := 0$, $1/d_1 = \varepsilon - \delta\, B$, and for $j \geq 1$,

$$1/d_{j+1} = \varepsilon - \delta\, B\,(1 + d_j + \, d_{j-1}\, d_j).$$

Furthermore, since $\delta\, B = \frac{\varepsilon^3}{16}$, by an elementary calculation, one gets that

$$\varepsilon^{-1} < d_j < 2\varepsilon^{-1} \quad \forall j \geq 1. \tag{2.5}$$

In particular d_j are positive and finite. It concludes for any $0 \leq k \leq n-1$ and for $0 < s < \delta$,

$$a_k \leq d_{k+1} ... d_n\, a_n \leq \frac{C}{\varepsilon^n}\, a_n.$$

It remains to verify (2.2) (as (2.1) is its consequence with the convention that $a_k = 0$ for $k < 0$). Indeed, since

$$\varepsilon\, \omega \leq \omega + dd^c[(1-\varepsilon)\psi + m(\varepsilon) + s] \quad \text{and} \quad U(\varepsilon, s) = \{\varphi < (1-\varepsilon)\psi + m(\varepsilon) + s\},$$

it follows from Lemma 2.1 that

$$\varepsilon \int_{U(\varepsilon,s)} \omega_\varphi^k \wedge \omega^{n-k} \leq \int_{U(\varepsilon,s)} \omega_{(1-\varepsilon)\psi} \wedge \omega_\varphi^k \wedge \omega^{n-k-1} \leq \int_{U(\varepsilon,s)} \omega_\varphi^{k+1} \wedge \omega^{n-k-1} + R,$$

where

$$R = \int_{U(\varepsilon,s)} [(1-\varepsilon)\psi + m(\varepsilon) + s - \varphi] dd^c \left(\omega_\varphi^k \wedge \omega^{n-k-1}\right) \leq s\, B\, (a_k + a_{k-1} + a_{k-2}).$$

Thus, for $0 < s < \delta = \frac{\varepsilon^3}{16B}$,

$$\varepsilon a_k \leq a_{k+1} + \delta\, B (a_k + a_{k-1} + a_{k-2}).$$

The theorem follows. □

3. The domination principle

Let Ω be a bounded open set in $\mathbb{C}^n$. The constant $B > 0$ is defined as in (0.1) for $\bar{\Omega}$. The next theorem is an analogue of the modified comparison principle for a bounded open set in $\mathbb{C}^n$.

Theorem 3.1. *Fix $0 < \varepsilon < 1$. Let $\varphi, \psi \in PSH(\omega) \cap L^\infty(\Omega)$ be such that $\liminf_{\zeta \to z \in \partial\Omega} (\varphi - \psi)(\zeta) \geq 0$. Suppose that $M = \sup_\Omega (\psi - \varphi) > 0$, and $\omega + dd^c \psi \geq \varepsilon\omega$ in Ω. Then, for any $0 < s < \varepsilon_0 := \min\{\frac{\varepsilon^n}{16B}, M\}$,*

$$\int_{\{\varphi < \psi - M + s\}} \omega_\psi^n \leq \left(1 + \frac{sB}{\varepsilon^n}\, C_n\right) \int_{\{\varphi < \psi - M + s\}} \omega_\varphi^n,$$

where C_n is a uniform constant depending only on n.

Proof. It is very similar to the proof of the modified comparison principle. The lemmata we need have now the following form.

Lemma 3.2. *Let $T := (\omega + dd^c v_1) \wedge ... \wedge (\omega + dd^c v_{n-1})$ with $v_1, ..., v_{n-1} \in PSH(\omega) \cap L^\infty(\Omega)$ be a positive current of bidegree $(n-1, n-1)$. Let $\varphi, \psi \in PSH(\omega) \cap L^\infty(\Omega)$. If $\liminf_{\zeta \to z \in \partial\Omega} (\varphi - \psi)(\zeta) \geq 0$, then*

$$\int_{\{\varphi < \psi\}} dd^c \psi \wedge T \leq \int_{\{\varphi < \psi\}} dd^c \varphi \wedge T + \int_{\{\varphi < \psi\}} (\psi - \varphi)\, dd^c T.$$

Lemma 3.3. *Let $\varphi, \psi \in PSH(\omega) \cap L^\infty(\Omega)$ be such that $\liminf_{\zeta \to z \in \partial\Omega} (\varphi - \psi)(\zeta) \geq 0$. Suppose that $B \sup_{\{\varphi < \psi\}} (\psi - \varphi) \leq 1$. Then,*

$$\int_{\{\varphi < \psi\}} \omega_\psi^n \leq \int_{\{\varphi < \psi\}} \omega_\varphi^n + B \sup_{\{\varphi < \psi\}} (\psi - \varphi) \left(C_n \sum_{k=0}^{n-1} \int_{\{\varphi < \psi\}} \omega_\varphi^k \wedge \omega^{n-k} \right),$$

where the constant C_n depends only on n.

Having those the proof goes exactly as the one of Theorem 2.3. □

As a consequence we obtain the domination principle.

Corollary 3.4. *Let Ω be a bounded open set in $\mathbb{C}^n$. Let $u, v \in PSH(\omega) \cap L^\infty(\Omega)$ be such that $\liminf_{\zeta \to z \in \partial\Omega} (u - v)(\zeta) \geq 0$. Suppose that $(\omega + dd^c u)^n \leq (\omega + dd^c v)^n$. Then $v \leq u$ in Ω.*

Proof. First, we may assume that $\liminf_{\zeta\to z\in\partial\Omega}(u-v)(\zeta)\geq 2\alpha>0$. Otherwise, replace u by $u+2\alpha$ and then let $\alpha\to 0$. Thus there is a relatively compact open set Ω' such that $u(z)\geq v(z)+\alpha$ for $z\in\Omega\setminus\Omega'$. By subtracting the same constant, we also assume that $u,v\leq 0$. We argue by contradiction. Suppose that $\{u<v\}$ is non empty. Since Ω is bounded, there is a strictly psh function $\rho\in C^2(\bar\Omega)$ such that $-C\leq\rho\leq 0$ in Ω, for some constant $0<C$. Since, u,v,ρ are bounded in Ω, then after multiplying ρ by a small positive constant we see that there exist $0<\varepsilon,\tau<<1/2$ such that

$$dd^c\rho\geq 2\,\varepsilon\,\omega,\quad (1-\tau)^{1/n}+(2\tau)^{1/n}\leq 1+\varepsilon,$$

and

$$\{u<(1-\tau)^{1/n}v+(2\tau)^{1/n}u+\rho\}\subset\subset\Omega$$

is non empty. Put $\hat v:=(1-\tau)^{1/n}v+(2\tau)^{1/n}u+\rho$. Since $\omega_v^n\geq\omega_u^n$, it follows that

$$\omega_{\hat v}^n\geq\left[(1-\tau)^{1/n}\omega_v+(2\tau)^{1/n}\omega_u\right]^n\geq(1-\tau)\,\omega_v^n+2\tau\,\omega_u^n\geq(1+\tau)\omega_u^n.$$

Thus,

$$\omega+dd^c\hat v\geq\varepsilon\omega\quad\text{ and }\quad\omega_{\hat v}^n\geq(1+\tau)\,\omega_u^n \tag{3.1}$$

in Ω. Let us denote by $U(s)$ the set $\{u<\hat v-M+s\}$ with $M=\sup_\Omega(\hat v-u)>0$. Then for any $0<s<M$,

$$U(s)\subset\subset\Omega\quad\text{ and }\quad\sup_{U(s)}\{(\hat v-M+s)-u\}=s.$$

It follows from (3.1) that the assumptions of Theorem 3.1 are fulfilled for $\varphi:=u$, $\psi:=\hat v-M+s$. Hence, for any $0<s<\epsilon_0=\min\{\frac{\varepsilon^n}{16B},M\}$,

$$0<\int_{U(s)}(\omega+dd^c\hat v)^n\leq\left(1+\frac{sB}{\varepsilon^n}\,C_n\right)\int_{U(s)}\omega_u^n.$$

Then using (3.1), we get for $0<s<\epsilon_0$

$$0<\tau\int_{U(s)}\omega_u^n\leq\frac{s\,BC_n}{\varepsilon^n}\int_{U(s)}\omega_u^n. \tag{3.2}$$

Therefore $0<\tau\leq\frac{s\,BC_n}{\varepsilon^n}$. This is impossible when $0<s$ is small enough. Thus, the proof follows. $\square$

4. The Dirichlet problem in a bounded domain in $\mathbb{C}^n$

Denote by β the standard Kähler form $dd^c\|z\|^2$ in $\mathbb{C}^n$ and by ω an arbitrary Hermitian form in $\mathbb{C}^n$. Let Ω be a bounded open set in $\mathbb{C}^n$. We write $L^p(\omega^n)$ for $L^p(\Omega,\omega^n)$ and consider the Dirichlet problem for the Monge-Ampère equation with the background metric ω. Given $0\leq f\in L^p(\omega^n)$, $p>1$, and $\phi\in C(\partial\Omega)$, we seek for a solution to

$$\begin{cases} u\in PSH(\omega)\cap C(\bar\Omega), & \\ (\omega+dd^cu)^n=f(z)\,\omega^n & \text{in }\Omega,\\ u=\phi & \text{on }\partial\Omega,\end{cases} \tag{4.1}$$

where the equality in the second line is understood in sense of currents.

From the domination principle above and the stability estimates [15] we get the following result.

Theorem 4.1. *Let Ω be a bounded open set in $\mathbb{C}^n$ and let $u, v \in PSH(\omega) \cap C(\bar{\Omega})$ be such that*

$$(\omega + dd^c u)^n = f\omega^n, \quad (\omega + dd^c v)^n = g\omega^n$$

with $0 \leq f, g \in L^p(\omega^n)$, $p > 1$. Then

$$\|u - v\|_{L^\infty(\bar{\Omega})} \leq \sup_{\partial\Omega} |u - v| + C\|f - g\|^{\frac{1}{n}}_{L^p(\omega^n)},$$

where C depends only on Ω, ω and p.

Proof. Suppose that $\Omega \subset B(0, R) =: B_R$ (the ball with the origin at 0 and radius $R > 0$). We write $\omega^n = h\beta^n$ in B_R, where $0 < h \in C^\infty(\bar{B}_R)$ and we extend f, g onto B_R by setting $f = g = 0$ on $B_R \setminus \Omega$. Therefore, $fh, gh \in L^p(B_R, \beta^n)$. By [15], there is a unique $w \in PSH(B_R) \cap C(\bar{B}_R)$ solving $(dd^c w)^n = |fh - gh|\beta^n$ with $w = 0$ on ∂B_R. The stability estimate for the complex Monge-Ampère equation proven in [15] says that

$$\|w\|_{L^\infty(\bar{B}_R)} \leq C_1\|fh - gh\|^{\frac{1}{n}}_{L^p(B_R, \beta^n)},$$

where C_1 depends only on Ω, p. Since

$$(\omega + dd^c(u + w))^n \geq \omega_u^n + (dd^c w)^n = f\omega^n + |fh - gh|\beta^n \geq g\omega^n,$$

and $w \leq 0$ in Ω, we can apply the domination principle for $\varphi := u + w$ and $\psi := v + \sup_{\partial\Omega} |u - v|$ to get that $u + w \leq v + \sup_{\partial\Omega} |u - v|$ in Ω. Hence,

$$w - \sup_{\partial\Omega} |u - v| \leq v - u.$$

Similarly, we obtain $v - u \leq -w + \sup_{\partial\Omega} |u - v|$. So

$$\begin{aligned} |u - v| \leq \|w\|_{L^\infty} + \sup_{\partial\Omega} |u - v| &\leq \sup_{\partial\Omega} |u - v| + C_1\|fh - gh\|^{\frac{1}{n}}_{L^p(B_R, \beta^n)} \\ &\leq \sup_{\partial\Omega} |u - v| + C\|f - g\|^{\frac{1}{n}}_{L^p(\Omega, \omega^n)}, \end{aligned}$$

where C depends on Ω, p and $\sup_{\bar{\Omega}} h$. □

Theorem 4.2. *In a C^∞ strictly pseudoconvex domain there exists a unique continuous solution to the Dirichlet problem* (4.1).

Proof. Suppose that $\phi_j \in C^\infty(\partial\Omega)$ converges uniformly to ϕ and a sequence of smooth functions $f_j > 0$ converges to f in $L^p(\omega^n)$. From Theorem 1.1 in [11], it follows that for each j there exists a unique smooth solution $u_j \in PSH(\omega)$ of the corresponding Dirichlet problem

$$\begin{cases} (\omega + dd^c v)^n & = f_j\omega^n \quad \text{in } \Omega, \\ v & = \phi_j \quad \text{on } \partial\Omega. \end{cases}$$

Hence, from Theorem 4.1 we get that the solutions u_j form a Cauchy sequence in $C(\bar{\Omega})$. Thus, they converge uniformly to u in $PSH(\omega) \cap C(\bar{\Omega})$. Therefore $\omega_{u_j}^n$ converge weakly to ω_u^n by Proposition 1.2. It means that u is a continuous solution to the Dirichlet problem (4.1). Moreover, by the domination principle (Corollary 3.4) this solution is unique. The proof is completed. □

5. Existence of continuous solutions on a compact Hermitian manifold

Let (X,ω) be a compact Hermitian manifold of complex dimension n. The constant $B>0$ in (0.1) is used throughout this section. We denote by C a generic positive constant depending only on n, B, which may vary from line to line. We use the notation $Vol_\omega(E) := \int_E \omega^n$ for any Borel set E, and write $L^p(\omega^n)$ for $L^p(X,\omega^n)$.

5.1. L^∞ a priori estimates.

We first show how the modified comparison principle coupled with pluripotential theory techniques leads to L^∞ *a priori* estimates. Recall that for a Borel set $E\subset X$

$$cap_\omega(E) := \sup\left\{\int_E (\omega+dd^c\rho)^n : \rho\in PSH(\omega),\, 0\le\rho\le 1\right\}.$$

Proposition 5.1 ([7], Corollary 2.4)**.** *There are a universal number $0<\alpha=\alpha(X,\omega)$ and a uniform constant $0<C=C(X,\omega)$ such that for any Borel subset $E\subset X$*

$$Vol_\omega(E)\le C\exp\left(\frac{-\alpha}{cap_\omega^{\frac{1}{n}}(E)}\right).$$

Consequently, by Hölder's inequality, for any $0\le f\in L^p(\omega^n)$, $p>1$,

$$\int_E f\omega^n \le C\|f\|_{L^p(\omega^n)}\exp\left(-\frac{\tilde\alpha}{cap_\omega^{\frac{1}{n}}(E)}\right),$$

where $\tilde\alpha=\alpha/q$, $1/p+1/q=1$.

Let $h:\mathbb{R}_+\to(0,\infty)$ be an increasing function such that

$$\int_1^\infty \frac{1}{x[h(x)]^{\frac{1}{n}}}\,dx<+\infty. \tag{5.1}$$

In particular, $\lim_{x\to\infty}h(x)=+\infty$. Such a function h is called *admissible.* If h is admissible, then so is $A\,h$ for any number $A>0$. Define

$$F_h(x)=\frac{x}{h(x^{-\frac{1}{n}})}.$$

For such F_h we consider the family of bounded ω-psh functions such that their Monge-Ampère measures satisfy

$$\int_E \omega_\varphi^n \le F_h(cap_\omega(E)), \tag{5.2}$$

for any Borel set $E\subset X$. It follows from Proposition 5.1 that

Corollary 5.2. *Let $\varphi\in PSH(\omega)\cap L^\infty(X)$. If $\omega_\varphi^n=f\,\omega^n$ for $0\le f\in L^p(\omega^n)$, $p>1$, then ω_φ^n satisfies* (5.2) *for the admissible function $h_p(x)=C\|f\|_{L^p(\omega^n)}^{-1}\exp(ax)$ with some universal number $a>0$.*

Our next theorem is a generalization of a priori estimates in [16], [17] from the Kähler setting to the Hermitian one.

Theorem 5.3. *Fix $0 < \varepsilon < 1$. Let $\varphi, \psi \in PSH(\omega) \cap L^\infty(X)$ be such that $\varphi \leq 0$, and $-1 \leq \psi \leq 0$. Set $m(\varepsilon) = \inf_X [\varphi - (1-\varepsilon)\psi]$, and $\varepsilon_0 := \frac{1}{3}\min\{\varepsilon^n, \frac{\varepsilon^3}{16B}, 4(1-\varepsilon)\varepsilon^n, 4(1-\varepsilon)\frac{\varepsilon^3}{16B}\}$. Suppose that ω_φ^n satisfies (5.2) for an admissible function h. Then, for $0 < D < \varepsilon_0$,*

$$D \leq \kappa \left[cap_\omega(U(\varepsilon, D)) \right],$$

where $U(\varepsilon, D) = \{\varphi < (1-\varepsilon)\psi + m(\varepsilon) + D\}$, and the function κ is defined on the interval $(0, cap_\omega(X))$ by the formula

$$\kappa\left(s^{-n}\right) = 4\, C_n \left\{ \frac{1}{[h(s)]^{\frac{1}{n}}} + \int_s^\infty \frac{dx}{x\,[h(x)]^{\frac{1}{n}}} \right\},$$

with a dimensional constant C_n.

The following lemma is the crucial step in the proof of the theorem. It is an estimate of the capacity of sublevel sets. The proof goes through in the Hermitian setting thanks to the modified comparison principle (Theorem 2.3).

Lemma 5.4. *Fix $0 < \varepsilon < 1$. Let $\varphi, \psi \in PSH(\omega) \cap L^\infty(X)$ be such that $-1 \leq \psi \leq 0$. Set $m(\varepsilon) = \inf_X [\varphi - (1-\varepsilon)\psi]$ and*

$$U(\varepsilon, s) := \{\varphi < (1-\varepsilon)\psi + m(\varepsilon) + s\}.$$

For any $0 < s, t \leq \frac{1}{3}\min\{\varepsilon^n, \frac{\varepsilon^3}{16B}\}$ (with B defined above) one has

$$[(1-\varepsilon)\, t]^n \, cap_\omega(U(\varepsilon, s)) \leq (1 + C) \int_{U(\varepsilon, s + 4(1-\varepsilon)\, t)} \omega_\varphi^n.$$

Proof. Let $\rho \in PSH(\omega)$ be such that $0 \leq \rho \leq 1$. It follows that

$$U(\varepsilon, s) \subset \{\varphi < (1-\varepsilon)\left[(1-t)\psi + t\,\rho\right] + m(\varepsilon) + s\}.$$

If we use the notation

$$m(\varepsilon, t) := \inf_X \left(\varphi - (1-\varepsilon)\left[(1-t)\psi + t\,\rho\right]\right),$$

then $m(\varepsilon, t) \leq m(\varepsilon) \leq m(\varepsilon, t) + 2(1-\varepsilon)\, t$. Hence,

$$\begin{aligned} U(\varepsilon, s) \subset V &:= \{\varphi < (1-\varepsilon)\left[(1-t)\psi + t\,\rho\right] + m(\varepsilon, t) + s + 2(1-\varepsilon)\, t\} \\ &\subset U(\varepsilon, s + 4(1-\varepsilon)\, t). \end{aligned}$$

Then, Theorem 2.3 gives

$$\begin{aligned} [(1-\varepsilon)t]^n \int_{U(\varepsilon,s)} (\omega + dd^c\rho)^n &\leq \int_V \left(\omega + (1-\varepsilon)\, dd^c\left[(1-t)\psi + t\,\rho\right]\right)^n \\ &\leq \left(1 + \frac{s + 2(1-\varepsilon)\, t}{\varepsilon^n} C\right) \int_V \omega_\varphi^n \\ &\leq (1 + C) \int_{U(\varepsilon, s + 4(1-\varepsilon)\, t)} \omega_\varphi^n. \end{aligned}$$

Thus the lemma follows. ∎

After rescaling t the statement of Lemma 5.4 may be rephrased

Remark 5.5. For any $0 < s \leq \frac{1}{3}\min\{\varepsilon^n, \frac{\varepsilon^3}{16B}\}$, $0 < t \leq \frac{4}{3}(1-\varepsilon)\min\{\varepsilon^n, \frac{\varepsilon^3}{16B}\}$ we have

$$t^n\, cap_\omega(U(\varepsilon, s)) \leq 4^n C \int_{U(\varepsilon, s+t)} \omega_\varphi^n,$$

where C is a dimensional constant.

The proof of Theorem 5.3. For $0 < s < \varepsilon_0$, define

$$a(s) := [cap_\omega(U(\varepsilon, s))]^{\frac{1}{n}} > 0,$$

and

$$g(x) = [h(x)]^{\frac{1}{n}}.$$

From Remark 5.5 and the property (5.2) we infer that for any $0 < s, t < \varepsilon_0$

$$t\, a(s) \leq C\, \frac{a(s+t)}{g\left(\frac{1}{a(s+t)}\right)},$$

where $C = 4\,(1 + C_n)^{1/n}$. We may assume $C = 1$ after muliplying g by an appropriate constant. Hence,

$$t \leq \frac{a(s+t)}{a(s)\, g\left(\frac{1}{a(s+t)}\right)}. \tag{5.3}$$

Let $0 < D < \varepsilon_0$. Applying (5.3) for $t := D - s$, and $0 < s < D$ we obtain

$$D - s \leq \frac{a(D)}{a(s) g(\frac{1}{a(D)})}.$$

Set

$$s_0 := \sup\{0 < s < D : a(D) > e\, a(s)\}.$$

Since $\lim_{t\to s^-} a(t) = a(s)$, so $s_0 < D$. It is clear that $a(D) \leq e\, a(s_0^+)$, where $a(s^+) = \lim_{t\to s^+} a(t)$. It follows that

$$D - s_0 \leq \lim_{s\to s_0^+} \frac{a(D)}{a(s) g(\frac{1}{a(D)})} = \frac{a(D)}{a(s_0^+) g(\frac{1}{a(D)})} \leq \frac{e}{g(\frac{1}{a(D)})}.$$

Thus, the theorem will follow if we have the estimate of s_0 from above. We define by induction a strictly decreasing sequence which begins with s_0, and for $j \geq 0$, satisfies

$$s_{j+1} := \sup\{0 < s < s_j : a(s_j) > e\, a(s)\}.$$

It follows that

$$a(s_j) \leq e\, a(s_{j+1}^+).$$

By monotonicity of $a(t)$ and the definition of s_{j+1} there exists $s_{j+2} < t < s_{j+1}$ such that

$$e\, a(s_{j+2}^+) \leq e\, a(t) < a(s_j).$$

Hence, we have

$$\frac{1}{e} a(s_{j+1}) \leq a(s_{j+2}^+) \leq \frac{1}{e} a(s_j). \tag{5.4}$$

We are ready to estimate s_0. Applying (5.3) for $s = s_{j+1}$, $t = s_j - s_{j+1}$ we have

$$s_j - s_{j+1} \leq \lim_{x\to s_{j+1}^+} \frac{a(s_j)}{a(x) g\left(\frac{1}{a(s_j)}\right)} = \frac{a(s_j)}{a(s_{j+1}^+) g\left(\frac{1}{a(s_j)}\right)}.$$

Then, using the first inequality in (5.4), we get

$$s_j - s_{j+1} \le \frac{e}{g\left(\frac{1}{a(s_j)}\right)}.$$

Setting

$$x_j := \frac{1}{a(s_j)},$$

we have, using the second inequality in (5.4) and $a(s_{j+2}) \le a(s_{j+2}^+)$,

$$\frac{x_{j+2} - x_j}{x_{j+2} g(x_{j+2})} = \frac{a(s_j) - a(s_{j+2})}{a(s_j)} \frac{1}{g\left(\frac{1}{a(s_{j+2})}\right)} \ge \frac{e-1}{e} \frac{1}{g\left(\frac{1}{a(s_{j+2})}\right)}.$$

Combining the last two estimates we have

$$s_{j+2} - s_{j+3} \le \frac{e^2}{e-1} \frac{x_{j+2} - x_j}{x_{j+2}\, g(x_{j+2})} \le \frac{e^2}{e-1} \int_{x_j}^{x_{j+2}} \frac{dx}{x\, g(x)},$$

as $xg(x)$ is an increasing function. Therefore,

$$s_2 = \sum_{j=0}^{\infty} (s_{j+2} - s_{j+3}) \le \frac{e^2}{e-1} \sum_{j=0}^{\infty} \int_{x_j}^{x_{j+2}} \frac{dx}{x\, g(x)}.$$

Since

$$\sum_{j=0}^{\infty} \int_{x_j}^{x_{j+2}} \frac{dx}{x\, g(x)} \le 2 \int_{x_0}^{\infty} \frac{dx}{x\, g(x)} \le 2 \int_{\frac{1}{a(D)}}^{\infty} \frac{dx}{x\, g(x)},$$

and

$$s_0 = (s_0 - s_1) + (s_1 - s_2) + s_2 \le \frac{2\, e}{g\left(\frac{1}{a(D)}\right)} + s_2,$$

we have

$$s_0 \le \frac{2\, e}{g\left(\frac{1}{a(D)}\right)} + \frac{2e^2}{e-1} \int_{\frac{1}{a(D)}}^{\infty} \frac{dx}{x\, g(x)}.$$

Consequently

$$D \le \frac{3e}{g\left(\frac{1}{a(D)}\right)} + \frac{2e^2}{e-1} \int_{\frac{1}{a(D)}}^{\infty} \frac{dx}{x\, g(x)}.$$

This is equivalent to the statement of the theorem. □

In the next step, we will see how Theorem 5.3 implies a L^∞ *a priori* bound on the solutions of the Monge-Ampère equation with the right hand side in L^p, $p > 1$.

Suppose that $\varphi \in PSH(\omega) \cap L^\infty(X)$, $\sup_X \varphi = 0$ satisfies

$$\omega_\varphi^n = f\omega^n, \tag{5.5}$$

where $0 \le f \in L^p(\omega^n)$, $p > 1$. Then, ω_φ^n satisfies (5.2) for $h(x) = C\|f\|_{L^p(\omega^n)}^{-1} \exp(ax)$, $a > 0$ (Corollary 5.2). Let $\hbar$ be the inverse function of κ in Theorem 5.3. Then, $\hbar$ is also an increasing function.

Corollary 5.6. *Let φ, f be as in (5.5). There exists a constant $0 < H = H(h)$, depending only on h, X, and ω such that*

$$-H \leq \varphi \leq 0. \tag{5.6}$$

Moreover, we have for $b \geq 1$,

$$H(b^{-n}h) \leq bH(h). \tag{5.7}$$

Proof. Applying Theorem 5.3 for $\psi = 0$, and $\varepsilon = 1/2$, we have

$$s \leq \kappa \left[cap_\omega \left(\{\varphi < \inf_X \varphi + s\} \right) \right] \Rightarrow \hbar(s) \leq cap_\omega \left(\{\varphi < \inf_X \varphi + s\} \right) \tag{5.8}$$

for $0 < s < \varepsilon_0$. Moreover, Proposition 2.5 in [7] says that

$$cap_\omega \left(\{\varphi < \inf_X \varphi + s\} \right) \leq \frac{C|||\varphi|||_{L^1(\omega^n)}}{|\inf_X \varphi + s|},$$

where C and

$$|||\varphi|||_{L^1(\omega^n)} := \sup \left\{ \int_X |\varphi|\omega^n : \varphi \in PSH(\omega), \sup_X \varphi = 0 \right\}$$

are uniform constants. Two last inequalities imply

$$\hbar(s) \leq \frac{C}{|\inf_X \varphi + s|} |||\varphi|||_{L^1(\omega^n)}.$$

Therefore,

$$|\inf_X \varphi| \leq s + \frac{C|||\varphi|||_{L^1(\omega^n)}}{\hbar(s)}$$

for $0 < s < \varepsilon_0$. This gives (5.6). In order to obtain (5.7), we proceed as follows. Let $\phi \in PSH(\omega) \cap L^\infty(X)$, $\sup_X \phi = 0$, be such that for any Borel set E

$$\int_E \omega_\phi^n \leq b^n F_h(cap_\omega(E)).$$

It follows from the formula for the function κ in Theorem 5.3 that the function κ' for $b^{-n}h$ is $b\kappa$. The above argument implies that

$$|\inf_X \phi| \leq s + \frac{C|||\varphi|||_{L^1(\omega^n)}}{\hbar(\frac{s}{b})},$$

where we used the fact that the inverse of $\kappa'(.) = b\kappa(.)$ is $\hbar'(.) = \hbar(\frac{1}{b}\ .)$. From the formula for the function κ associated to the admissible function $h(x) = C\exp(ax)$, $a > 0$, it follows that for $b \geq 1$, $0 < x < \varepsilon_0$,

$$b\hbar(\frac{x}{b}) \geq \hbar(x).$$

Thus, for $0 < s < \varepsilon_0$,

$$|\inf_X \phi| \leq b \left(\frac{s}{b} + \frac{C|||\varphi|||_{L^1(\omega^n)}}{b\hbar(\frac{s}{b})} \right) \leq b \left(s + \frac{C|||\varphi|||_{L^1(\omega^n)}}{\hbar(s)} \right).$$

The corollary follows. □

5.2. **Weak solutions to the complex Monge-Ampère equation.** In this section we are going to study the existence of weak solutions for the Monge-Ampère equation on X. Let $0 \leq f \in L^p(\omega^n)$, $p > 1$. We wish to solve the equation

$$\begin{cases} u \in PSH(X,\omega) \cap L^\infty(X), \\ (\omega + dd^c u)^n = f\,\omega^n. \end{cases} \tag{5.9}$$

In general ω is not closed, and then the appropriate statement of (5.9) is that there exist a constant $c > 0$ and a bounded (or continuous) ω-psh function u such that

$$(\omega + dd^c u)^n = c\, f\,\omega^n. \tag{5.10}$$

Remark 5.7. If $f \equiv 0$, then the equation (5.10) has no bounded solution.

Proof. It is a immediate consequence of the inequality (5.14) below as an open subset has a positive capacity. □

Theorem 5.8. *Let $0 \leq f \in L^p(\omega^n)$, $p > 1$, be such that $\int_X f\omega^n > 0$. There exist a constant $c > 0$ and $u \in PSH(\omega) \cap C(X)$ satisfying the equation* (5.10).

Proof. Choose $f_j \in L^p(\omega^n)$ smooth, strictly positive and converging to f in $L^p(\omega^n)$. By a theorem of Tosatti and Weinkove [22], for each $j \geq 1$, there exist a unique $u_j \in PSH(\omega) \cap C^\infty(X)$ with $\sup_X u_j = 0$ and a unique constant $c_j > 0$ such that

$$(\omega + dd^c u_j)^n = c_j\, f_j\,\omega^n. \tag{5.11}$$

Lemma 5.9. *The sequence $\{c_j\}$ is bounded away from* 0 *and bounded from above. In particular, the family $\{c_j f_j\}$ is bounded in $L^p(\omega^n)$.*

Proof. We first show that c_j's are uniformly bounded from above. Since $f_j \to f$ in $L^1(\omega^n)$, we also have $f_j^{\frac{1}{n}} \to f^{\frac{1}{n}}$ in $L^1(\omega^n)$. Because $\int_X f\omega^n > 0$, $\int_X f^{\frac{1}{n}}\omega^n > 0$ one obtains

$$\int_X f_j^{\frac{1}{n}}\omega^n > \frac{\int_X f^{\frac{1}{n}}\omega^n}{2} > 0$$

for $j > j_0$ ($j_0 \geq 1$ depends on f). The pointwise arithmetic-geometric means inequality implies that

$$(\omega + dd^c u_j)\wedge\omega^{n-1} \geq \left[\frac{(\omega + dd^c u_j)^n}{\omega^n}\right]^{\frac{1}{n}}\omega^n = (c_j\, f_j)^{\frac{1}{n}}\omega^n.$$

Hence,

$$c_j^{\frac{1}{n}}\int_X f_j^{\frac{1}{n}}\omega^n \leq \int_X (\omega + dd^c u_j)\wedge\omega^{n-1}.$$

It follows that for $j > j_0$,

$$c_j^{\frac{1}{n}} \leq \frac{2}{\int_X f^{\frac{1}{n}}\omega^n}\int_X (\omega + dd^c u_j)\wedge\omega^{n-1}. \tag{5.12}$$

To end the proof we need to show that the right hand side is uniformly bounded from above. Since $\sup_X u_j = 0$, it follows that

$$\int_X |u_j|\omega^n \leq C_1, \tag{5.13}$$

with a uniform constant C_1 (see e.g. [7], Proposition 2.5). Hence, using the Stokes theorem we have

$$\begin{aligned}\int_X dd^c u_j \wedge \omega^{n-1} &= \int_X u_j \wedge dd^c(\omega^{n-1}) \\ &\leq B \int_X |u_j| \omega^n \\ &\leq B\, C_1.\end{aligned}$$

Combining this with (5.12) we conclude that $\{c_j\}$ is bounded from above.

It remains to verify that $\{c_j\}$ is bounded away from 0. Applying Remark 5.5 for $\varepsilon = 1/2$, $\psi = 0$ and $0 \geq \varphi \in PSH(\omega) \cap L^\infty(X)$, with $S = \inf_X \varphi$, we get for $0 < s, t < \varepsilon_0$,

$$t^n \, cap_\omega(\varphi < S + s) \leq C \int_{\{\varphi < S+s+t\}} \omega_\varphi^n. \tag{5.14}$$

From Remark 5.5 for $\varphi := u_j$ with $\inf_X u_j = S_j$ and the Hölder inequality, it follows that, for $0 < s, t < \varepsilon_0$,

$$\begin{aligned} t^n cap_\omega(u_j < S_j + s) &\leq C \int_{\{u_j < S_j + s + t\}} c_j \, f_j \omega^n \\ &\leq C\, c_j \|f_j\|_{L^p(\omega^n)} [Vol_\omega(\{u_j < S_j + s + t\})]^{\frac{1}{q}},\end{aligned}$$

where $1/p + 1/q = 1$. Therefore, for fixed $0 < s = t < \varepsilon_0$,

$$cap_\omega(u_j < S_j + s) \leq s^{-n} C\, c_j \|f_j\|_{L^p(\omega^n)} [Vol_\omega(\{u_j < S_j + 2s\})]^{\frac{1}{q}} := c_j \, C_1 s^{-n},$$

with C_1 depending also on X and f. From Theorem 5.3 we know that

$$s \leq \kappa(cap_\omega(\{u_j < S_j + s\})) \leq \kappa(c_j \, C_1 s^{-n}).$$

Since $\lim_{x \to 0^+} \kappa(x) = 0$, c_j must be uniformly bounded away from 0. □

We proceed to finish the proof of the theorem.

Uniform bound of $||u_j||_{L^\infty}$. Since $c_j \, f_j$ are uniformly bounded in $L^p(\omega^n)$, the L^∞ a priori estimate from [7] (or Corollary 5.6) implies that $\{u_j\}$ are uniformly bounded. Thus there exists $H > 0$ such that $-H \leq u_j \leq 0$ for every j. By rescaling we may assume from now on that $H = 1$. Moreover, by passing to a subsequence, it is assumed that $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^1(\omega^n)$ and $0 < c = \lim c_j$.

The existence of a continuous solution. Let us use the notation

$$S_{kj} := \inf_X (u_k - u_j) \leq 0, \qquad M_{kj} := \sup_X (u_k - u_j) \geq 0.$$

We are going to show that both $S_{kj} \to 0$ and $M_{kj} \to 0$ as $k, j \to +\infty$, arguing by contradiction. Suppose that S_{kj} does not converge to 0 as $k, j \to +\infty$. So there exists $0 < \varepsilon < 1$ such that

$$S_{kj} \leq -4\,\varepsilon,$$

for arbitrarily large $k \neq j$. In order to simplify the notation, we write $\varphi := u_k$, $\psi := u_j$, and $S := S_{kj}$, for such a pair j, k. Put

$$m(\varepsilon) := \inf_X \left[\varphi - (1 - \varepsilon)\psi\right].$$

It follows that

$$m(\varepsilon) \leq S.$$

Applying Remark 5.5 we have, for any $0 < s, t < \varepsilon_0$ (see the definition in Theorem 5.3),

$$t^n \, cap_\omega(U(\varepsilon, s)) \leq C \int_{U(\varepsilon, s+t)} \omega_\varphi^n, \tag{5.15}$$

where

$$U(\varepsilon, s) := \{\varphi < (1-\varepsilon)\psi + m(\varepsilon) + s\}.$$

Since $-1 \leq \varphi, \psi \leq 0$, $S < -4\varepsilon$, $0 < s, t < \varepsilon_0$, we have the following inclusions

$$U(\varepsilon, s+t) \subset \{\varphi < \psi + S + \varepsilon + s + t\} \subset \{\varphi < \psi - \varepsilon\} \subset \{|\varphi - \psi| > \varepsilon\}.$$

Thus, from (5.15) and Hölder's inequality, with $1/p + 1/q = 1$, we get that

$$\begin{aligned} t^n \, cap_\omega(U(\varepsilon, s)) &\leq C \int_{\{|\varphi-\psi|>\varepsilon\}} \omega_\varphi^n \\ &\leq C \|c_k f_k\|_{L^p(\omega^n)} \left[Vol_\omega(\{|\varphi - \psi| > \varepsilon\})\right]^{\frac{1}{q}} \end{aligned}$$

(since $\omega_\varphi^n = c_k f_k \omega^n$). We have already seen that $c_k \, f_k$ is uniformly bounded in $L^p(\omega^n)$. Hence, for fixed $0 < s = t = D < \varepsilon_0$,

$$\begin{aligned} cap_\omega(U(\varepsilon, D)) &\leq D^{-n} C(n) \|c_k \, f_k\|_{L^p(\omega^n)} \left[Vol_\omega(\{|\varphi - \psi| > \varepsilon\})\right]^{\frac{1}{q}} \\ &\leq C_2 \left[Vol_\omega(\{|\varphi - \psi| > \varepsilon\})\right]^{\frac{1}{q}}, \end{aligned}$$

where C_2 is a constant independent of j, k. Next, we apply Theorem 5.3, after taking values of κ of both sides of the above inequality

$$D \leq \kappa \left[cap_\omega(U(\varepsilon, D))\right] \leq \kappa \left[C_2 \left[Vol_\omega(\{|\varphi - \psi| > \varepsilon\})\right]^{\frac{1}{q}}\right].$$

This leads to a contradiction because $\lim_{x\to 0^+} \kappa(x) = 0$, and

$$Vol_\omega(\{|\varphi - \psi| > \varepsilon\}) = Vol_\omega(\{|u_k - u_j| > \varepsilon\}) \to 0 \qquad \text{as} \quad k, j \to +\infty.$$

Thus, $S_{kj} \to 0$ as $k, j \to +\infty$. Also $M_{kj} \to 0$ as $k, j \to +\infty$ since $M_{kj} = -S_{jk}$. Hence,

$$|u_k - u_j| \leq |S_{kj}| + |M_{kj}| \to 0 \quad \text{as} \quad k, j \to +\infty.$$

We conclude that $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $PSH(\omega) \cap C(X)$. Let u and c be the limit points of $\{u_j\}$ and $\{c_j\}$ respectively. Then the continuous function $u \in PSH(\omega) \cap C(X)$ solves

$$(\omega + dd^c u)^n = c \, f \, \omega^n,$$

in the weak sense of currents. □

It is worth to record here that from the above argument we get a weak stability statement.

Corollary 5.10. *Let $\{u_j\}_{j=1}^\infty \subset PSH(\omega) \cap C(X)$ be such that $\sup_X u_j = 0$. Suppose that for every $j \geq 1$,*

$$\omega_{u_j}^n = f_j \omega^n,$$

where f_j's are uniformly bounded in $L^p(\omega^n)$, $p > 1$. If $\{u_j\}$ is Cauchy in $L^1(\omega^n)$, then it is Cauchy in $PSH(\omega) \cap C(X)$.

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