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A TOPOLOGICAL APPROACH TO THE ALGORITHMIC COMPUTATION OF THE CONLEY INDEX OF POINCARÉ MAPS

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ABSTRACT. A new algorithm for computing the Conley index of the Poincaré map of a time-periodic non-autonomous ordinary differential equation is presented. The algorithm is based on a theorem which reduces the computation of the index to the study of certain singular chains on an index pair for some small-step translation operator of the equation. In particular, no numerical enclosures of the Poincaré map are required. Concrete numerical examples for planar systems are provided.

1. INTRODUCTION

1.1. **Motivation.** Differential equations constitute one of the fundamental tools of contemporary science and technology. Since in most cases explicit formulas for solutions are not available, usually qualitative analytic or quantitative numerical methods are used to study the flow induced by a differential equation. Both approaches have their limitations. Analytic methods are rigorous but of limited scope and often too complex for concrete problems of practical interest. Numerical methods are approximate but relatively straightforward to apply and, therefore, broadly used. Unfortunately, numerical errors may significantly falsify the information by introducing spurious or ghost solutions. These are approximate solutions which exhibit some qualitative features such as periodic or chaotic behavior but the reason for this behavior does not lie in the differential equation but in the numerical method. The existence of such examples has been well documented in the literature (see [8, 10, 9] or [29] and the references within). This introduces a level of uncertainty which may be not acceptable in some safety critical applications such as flight control or nuclear power plant design.

In the last twenty years there have been several successful attempts to combine the qualitative analytic methods with quantitative numerical methods in the form of computer assisted proofs consisting of algorithmic qualitative analysis based on rigorous numerical methods. This approach brought many significant achievements [14, 32, 1, 2, 33, 3, 13, 25]. Although computer assisted proofs require substantial computational power, which limits the method today, the rapid growth of computer technology together with the ongoing progress in the development of new algorithms makes them a likely method in the future. In this paper we go in this direction and present theory and algorithms for efficient computation of the Conley index of isolated invariant sets for Poincaré maps, a crucial ingredient of several computer assisted proofs.

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1.2. **Background.** The Conley index is a topological invariant frequently used in computer assisted proofs in dynamics. It is defined for isolated invariant sets of flows via so called isolating blocks [4, 5] and for discrete dynamical systems via index pairs [18]. Roughly speaking, an index pair of an isolated invariant set S of a discrete dynamical system generated by a map f is a pair (N,L) of compact sets such that $N \setminus L$ is a neighborhood of S and L is positively invariant with respect to N. It follows that f generates a continuous map $f_{(N,L)}$ of the quotient space N/L, hence also an endomorphism $H(f_{(N,L)})$ of the homology $H(N/L,*)$. Then, the reduction by the Leray functor, which in the finite dimensional case consists in quotienting out the union of the kernels of iterates of $H(f_{(N,L)}))$, leads to an automorphism $RH(f_{(N,L)})$ of some vector space. We call the conjugacy class of $RH(f_{(N,L)})$ the Conley index of S and denote it $\mathrm{CH(S,f)}$. In particular, if the Lefschetz number of $\mathrm{CH(S,f)}$ is nonzero, then f has a fixed point in S.

So far, no useful algorithms constructing isolating blocks for flows are known, but algorithms constructing index pairs for discrete dynamical systems are available and reasonably efficient [31, 11, 20]. On the theoretical side, the Conley index for a flow is in one-to-one correspondence with the Conley index for the h-translation operator of the flow [17]. The algorithmic version of the result is less general [23, 24] and requires some caution [15] but may be used fruitfully to compute algorithmically the Conley index for flows by applying an algorithm for maps. As we explain in the overview and show in detail in the paper, a similar result may be used to reduce the computation of the Conley index of a Poincaré map to the study of an index pair of the h-translation operator of the flow.

Recall that the Poincaré map is defined on a hypersurface transversal to the solutions of the differential equation, the so called Poincaré section. It sends a point on the section to its first return time to the section along the trajectories of the flow. Periodic trajectories of the differential equation transversal to the hypersurface are in one-to-one correspondence with the fixed points of the Poincaré map. Thus, understanding the dynamics of the Poincaré map is crucial in understanding the dynamics of the differential equation, in particular in locating periodic orbits or establishing chaotic behavior. In particular, whenever the Conley index of a Poincaré map detects a fixed point, it also detects a periodic solution of the differential equation.

The computation of the Conley index may be achieved algorithmically if good, rigorous combinatorial enclosures of the values of the map are available. However, in the case of a Poincaré map such enclosures are very expensive to obtain, because they require long time rigorous numerical integration of the differential equations and the size of the enclosures grows exponentially with the integration time. In case of differential equations with strong expansion this is often prohibitive.

In this paper we restrict the study to the case of a T-periodic, non-autonomous ordinary differential equation. We treat it as an autonomous differential equation in the extended phase space Ω. It induces a differential equation on the cylinder Σ obtained by identifying points in Ω whose time coordinates differ by T. In practice, $\Omega = \mathbb{R} \times \mathbb{R}^n$ and $\Sigma = S^1 \times \mathbb{R}^n$, where the circle S^1 is equal to $\mathbb{R}/T\mathbb{Z}$. Then, for every hyperplane H in Ω orthogonal to the time axis its projection H_Σ onto Σ is a Poincaré section and the corresponding Poincaré map is just the T-translation along the trajectories.

1.3. **Overview.** In this brief overview we present the main results of the paper. We concentrate on their applicability and skip some technical assumptions, irrelevant for the general idea. We begin with the construction of an index pair (N,L) of an isolated invariant set S for the time-h discretization of the system on Σ with h substantially smaller than T in order to avoid long time integration. In the construction we use the algorithm for index pairs introduced in [20]. Put $H = 0\times\mathbb{R}^n$. The intersection $S_0 := S\cap H_\Sigma$ is then an isolated invariant set for the Poincaré map, as we show in Proposition 4.1. Unfortunately, the intersection $(N_0,L_0) := (N\cap H_\Sigma, L\cap H_\Sigma)$ is not necessarily an index pair for S_0. But, our main result, Theorem 4.5, claims that the Conley index of the Poincaré map can be obtained from (N,L) and (N_0,L_0) in the following way. We lift the pair (N,L) to $(\widetilde{N},\widetilde{L})$ in the extended phase space Ω. Given a relative chain u in (N_0,L_0) we construct a chain w in $\widetilde{N}\cap[0,T]\times\mathbb{R}^n$ whose boundary is the sum of the lift of u, the lift of some chain v in (N_0,L_0), and a chain in $\widetilde{L}\cap[0,T]\times\mathbb{R}^n$. By picking up a collection of chains u_i generating a basis of the homology of (N_0,L_0) and decomposing each homology class of the corresponding v_i on this basis we obtain a matrix A whose i-th column is the vector of coefficients of v_i in the decomposition. We prove that, up to conjugacy, the homological Conley index of the Poincaré map is just the Leray reduction of A. We provide an algorithm constructing the matrix A by pushing forward the chains u_i (Algorithm 6). The computation of the Leray reduction is then just elementary linear algebra.

The novelty and strength of the proposed method lies in replacing the very expensive and often prohibitive computations of the combinatorial enclosure of the Poincaré map by purely combinatorial computations of the chains v_i from the chains u_i.

The method presented in this paper has its roots in a theoretical result [28] concerning isolating segments, a generalization of isolating blocks. An algorithmic version of the results of [28] has been presented in [21]. The method of [21] provides results but is painful in implementation, because each individual case requires manual construction of isolating segments. Roughly speaking, this is related to the fact that verifying algorithmically that a given polytope is an isolating block is possible, but we have no useful algorithms constructing isolating blocks.

The rest of the paper is organized as follows. Section 2 contains preliminaries. Section 3 introduces isolated invariant sets and the Conley index for flows. Section 4 presents the main theoretical result of the paper, i.e., Theorem 4.5 showing how the computation of the Conley index of a Poincaré map may be reduced to the study of the time h-translation along the trajectories of the flow. Section 5 provides the proof of the main theoretical result. Section 6 recalls the main ideas concerning the rigorous numerics of dynamical systems. Section 7 discusses the algorithm based on Theorem 4.5. Section 8 presents the results of numerical experiments based on the algorithms of Section 7.

2. PRELIMINARIES

2.1. **Notation.** Let $k = 1,2,\ldots$. For a map $f\colon X\to X$ we denote by f^k its k-th iterate, i.e.,

$$f^k := f\circ\ldots\circ f \ (k \text{ times}).$$

We put also $f^0 :=$ identity and, if f is injective, $f^{-k} := (f^{-1})^k$.

In this paper, $\mathbb{F}$ denotes a fixed field. In the sequel the notion "vector space" or "linear space" refers to a vector space over $\mathbb{F}$. We write $V\cong W$ if the vector spaces V and W are isomorphic. If $S\subset V$, by $\mathrm{Lin}(S)$ we denote the subspace of V spanned by S. Let $A = [a_{ij}]_{i,j=1,\ldots,k}$ be a $(k\times k)$-matrix over $\mathbb{F}$ and $u = (u_1,\ldots,u_k)$ be an ordered basis of a vector space V. By A_u we denote the linear endomorphism $V\to V$ uniquely determined by

$$A_u(u_j) = \sum_i a_{ij}u_i.$$

In case $V = \mathbb{F}^k$ and e is the canonical basis of $\mathbb{F}^k$, we identify A with A_e.

By a *graded vector space* we mean a sequence $V = \{V_n\}_{n\in\mathbb{Z}}$ of vector spaces. We identify V with the direct sum of V_n, i.e.,

$$V = \bigoplus_{n\in\mathbb{Z}} V_n.$$

The graded vector space V is of *finite type* if V_n is finite-dimensional for all n and $V_n = 0$ for all $n < 0$ as well as for almost all $n \geq 0$.

By a graded linear map between graded vector spaces V and W we mean a sequence $\phi := \{\phi_n\}$ of linear maps $\phi_n \colon V_n \to W_n$. We identify ϕ with the direct sum of maps ϕ_n restricted to $\bigoplus V_n \to \bigoplus W_n$.

Let q and k_n, where $n = 0, \ldots, q$, be nonnegative integers. Put $k := \sum_{n=0}^{q} k_n$. By a *graded* $(k_0,\ldots,k_q)$*-matrix* (or, less precisely, a *graded k-matrix*) we mean a square k-matrix A of the diagonal-block form

$$A = \operatorname{diag}(A(0),\ldots,A(q)) := \begin{bmatrix} A(0) & 0 & & \\ 0 & A(1) & & \\ & & \ddots & \\ & & & A(q) \end{bmatrix},$$

where each $A(n)$ is a square k_n-matrix. We treat such a matrix as a graded endomorphism of the graded vector space of finite type

$$\mathbb{F}^k = \bigoplus_{n\in\mathbb{Z}} \mathbb{F}^{k_n},$$

where $k_n = 0$ if $n > q$ or $n < 0$.

Assume V is of finite type and let ϕ be a graded linear endomorphism $V \to V$. The *Lefschetz number* of ϕ is defined as

$$\Lambda(\phi) := \sum_{n=0}^{\infty} (-1)^n \operatorname{trace} \phi_n.$$

Assume $V_n = 0$ for $n > q$ and for $n < 0$. For each $n = 0, \ldots, q$ let $v(n)$ be a basis of V_n and let $A(n) = [a(n)_{ij}]$ be the matrix of ϕ_n with respect to $v(n)$. Then

$$\Lambda(\phi) = \Lambda(A) = \sum_{n=0}^{q} (-1)^n \sum_{i=0}^{\dim V_n} a(n)_{ii}.$$

2.2. **Retractors.** Denote by **Vect** the category of vector spaces over $\mathbb{F}$ and let **End** be the category of endomorphisms of **Vect**. Recall that the objects of **End** are pairs (V,α) where $V \in$ **Vect** and $\alpha \colon V \to V$ is an endomorphism in **Vect**. A morphism in **End** from (V,α) to (W,β) is a linear map $f \colon V \to W$ such that $f \circ \alpha = \beta \circ f$. We denote this morphism by $\vec{f}_{\alpha\beta} \colon (V,\alpha) \to (W,\beta)$. We refer to f as the underlying map of $\vec{f}_{\alpha\beta}$. Note that two different morphisms in **End** may have the same underlying map, hence we omit the index $\alpha\beta$ in the notation $\vec{f}_{\alpha\beta}$ only if α and β are really clear from the context.

The *forgetful functor* $F \colon$ **End** $\to$ **Vect** is defined by

$$\begin{array}{rcl} (V,\alpha) & \mapsto & V, \\ \vec{f}_{\alpha\beta} & \mapsto & f. \end{array}$$

The following proposition may be easily verified.

Proposition 2.1. *Let* (V,α) *and* (W,β) *be objects of* **End**.

(i) If $(V,\alpha) \in$ **End**, *then* $\vec{\alpha}_{\alpha\alpha} \in$ **End**$((V,\alpha),(V,\alpha))$.

(ii) Let $f\colon V \to W$ be a morphism in **Vect***, i.e., a homomorphism of vector spaces, and assume that it induces a morphism $\vec{f}_{\alpha\beta}\colon (V,\alpha) \to (W,\beta)$ in* **End***. Then $\vec{f}_{\alpha\beta}$ is an isomorphism (respectively epimorphism, monomorphism) in* **End** *if and only if $f = F\vec{f}_{\alpha\beta}$ is an isomorphism (respectively epimorphism, monomorphism) in* **Vect***.*

Let **Aut** denote the category of automorphisms of **Vect**. Recall that **Aut** is the full subcategory of **End** whose objects are pairs (V,α) such that $\alpha\colon V \to V$ is an automorphism in **Vect**.

Let $R\colon$ **End** $\to$ **Aut** be a covariant functor. Then, given an object $(V,\alpha) \in$ **End**, its associated object in **Aut** is $R(V,\alpha) = (R_o(V,\alpha), R_a(V,\alpha))$, where $R_o(V,\alpha)$ is an object in **Vect** and $R_a(V,\alpha)$ is an automorphism of $R_o(V,\alpha)$. We say that R is a *retractor* if R is identity on **Aut**. By Proposition 2.1(i), given (V,α) in **End**, we have a morphism $\vec{\alpha}_{\alpha\alpha}$ in **End**. We say that R is *normal*, if for any object (V,α) in **End** the underlying map of $R\vec{\alpha}_{\alpha\alpha}$ coincides with $R_a(V,\alpha)$, that is if $FR\vec{\alpha}_{\alpha\alpha} = R_a(V,\alpha)$.

Let (V,α), (W,β) be two objects of **End**. Then, for any $r,s \geq 1$ we have objects (V,α^r), (W,β^s) in **End**. If for some $f\colon V \to W$ we have $\vec{f}_{\alpha\beta}\colon (V,\alpha) \to (W,\beta)$ and $\vec{f}_{\alpha^r\beta^s}: (V,\alpha^r) \to (W,\beta^s)$ as morphisms in **End**, then in general they are different. Hence, also the morphisms $R\vec{f}_{\alpha\beta}$ and $R\vec{f}_{\alpha^r\beta^s}$ or even their underlying maps may be different. We say that R is *consistent* if for any $f\colon V \to W$ such that $\vec{f}_{\alpha\beta}\colon (V,\alpha) \to (W,\beta)$ and any $r,s \geq 1$ such that $\vec{f}_{\alpha^r\beta^s}\colon (V,\alpha^r) \to (W,\beta^s)$ we have $FR\vec{f}_{\alpha\beta} = FR\vec{f}_{\alpha^r\beta^s}$, i.e., if the underlying linear maps of $R\vec{f}_{\alpha\beta}$ and $R\vec{f}_{\alpha^r\beta^s}$ coincide.

The simplest example of a normal, consistent retractor is the Leray functor [19] defined as follows. For an object (V,α) in **End**, let $\operatorname{gker}(\alpha) = \cup_{n\in\mathbb{N}} \ker(\alpha^n)$ and $\overline{V} = V/\operatorname{gker}(\alpha)$. This yields an induced map $\overline{\alpha}\colon \overline{V} \to \overline{V}$, which is injective. Then let $\overline{\overline{V}} = \operatorname{gim}(\alpha) := \cap_{n\in\mathbb{N}} \operatorname{im}(\overline{\alpha}^n)$ and let $R(V,\alpha) = (\overline{\overline{V}}, \overline{\alpha}\restriction_{\overline{\overline{V}}})$. A map $f\colon V \to W$ then induces a map $\overline{\overline{V}} \to \overline{\overline{W}}$ which describes $\vec{f}$. The normality of the Leray functor is immediate and consistency follows because for any linear endomorphism γ and $n \geq 1$: $\operatorname{gker}(\gamma) = \operatorname{gker}(\gamma^n)$ and $\operatorname{gim}(\gamma) = \operatorname{gim}(\gamma^n)$. One can show that also the direct limit functor and the inverse limit functor [19] are normal, consistent retractors.

Proposition 2.2. *Assume R is a normal, consistent retractor. Let $\alpha\colon V \to V$ and $\beta\colon W \to W$ be endomorphisms in* **Vect** *and $f\colon V \to W$ be a linear map in* **Vect** *such that $\vec{f}_{\alpha\beta}\colon (V,\alpha) \to (W,\beta)$ is a morphism in* **End***. Assume there are integers $r,s \geq 1$ and a linear map $g\colon W \to V$, such that the diagram*

(1)
$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\alpha^r"'] & W \arrow[d, "\beta^s"] \arrow[dl, "g"'] \\
V \arrow[r, "f"] & W
\end{tikzcd}$$

commutes. Then $R\vec{f}_{\alpha\beta}\colon R(V,\alpha) \to R(W,\beta)$ is an isomorphism.

Proof. Observe that diagram (1) yields the following two commutative diagrams in **End** where the right one is built from the left one by applying the functor R.

$$\begin{tikzcd}
(V,\alpha^r) \arrow[r, "\vec{f}_{\alpha^r\beta^s}"] \arrow[d, "\vec{\alpha}^r{}_{\alpha^r\alpha^r}"'] & (W,\beta^s) \arrow[d, "\vec{\beta}^s{}_{\beta^s\beta^s}"] \arrow[dl, "\vec{g}_{\beta^s\alpha^r}"'] \\
(V,\alpha^r) \arrow[r, "\vec{f}_{\alpha^r\beta^s}"] & (W,\beta^s)
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
R(V,\alpha^r) \arrow[r, "R\vec{f}_{\alpha^r\beta^s}"] \arrow[d, "R\vec{\alpha}^r{}_{\alpha^r\alpha^r}"'] & R(W,\beta^s) \arrow[d, "R\vec{\beta}^s{}_{\beta^s\beta^s}"] \arrow[dl, "R\vec{g}_{\beta^s\alpha^r}"'] \\
R(V,\alpha^r) \arrow[r, "R\vec{f}_{\alpha^r\beta^s}"] & R(W,\beta^s)
\end{tikzcd}$$

Applying the forgetful functor to the right diagram, we obtain the following commutative diagram in **Vect**.

$$\begin{tikzcd}
FR(V,\alpha^r) \arrow[r, "FR\vec{f}_{\alpha^r\beta^s}"] \arrow[d, "FR\vec{\alpha}^r{}_{\alpha^r\alpha^r}"'] & FR(W,\beta^s) \arrow[d, "FR\vec{\beta}^s{}_{\beta^s\beta^s}"] \arrow[dl, "FR\vec{g}_{\beta^s\alpha^r}"'] \\
FR(V,\alpha^r) \arrow[r, "FR\vec{f}_{\alpha^r\beta^s}"] & FR(W,\beta^s)
\end{tikzcd}$$

Since R is a normal rectractor, the morphisms $FR\vec{\alpha}^r{}_{\alpha^r\alpha^r}$ and $FR\vec{\beta}^s{}_{\beta^s\beta^s}$ are automorphisms in **Vect**. Hence, it follows from the commutativity of the upper triangle that $FR\vec{f}_{\alpha^r\beta^s}$ is a monomorphism in **Vect**, because $FR\vec{\alpha}^r{}_{\alpha^r\alpha^r}$ is an automorphism. Similarly, the commutativity of the lower triangle implies that $FR\vec{f}_{\alpha^r\beta^s}$ is an epimorphism, because $FR\vec{\beta}^s{}_{\beta^s\beta^s}$ is an automorphism. Hence, $FR\vec{f}_{\alpha^r\beta^s}$ is an isomorphism in **Vect**. Since R is consistent, also $FR\vec{f}_{\alpha\beta}$ is an isomorphism in **Vect**. Thus, by Proposition 2.1(ii) and since $R\vec{f}_{\alpha\beta}$ is a morphism in **End**, we conclude that $R\vec{f}_{\alpha\beta}$ is an isomorphism in **End**. □

2.3. **Topology.** Let (X,A) be a topological pair and let $*$ be a point outside of X. Define a set

$$X/A := (X\setminus A)\cup\{*\}$$

and the *quotient map* $q\colon X\cup\{*\}\to X/A$,

$$q(x) := \begin{cases} x & \text{if } x\in X\setminus A, \\ * & \text{if } x\in A \text{ or } x=*. \end{cases}$$

In particular $X/\varnothing = X\cup\{*\}$. Usually, we write $[x]$ instead of $q(x)$. Endow X/A with the quotient topology, i.e., U is open in X/A if and only if $q^{-1}(U)$ is open in $X\cup\{*\}$ with the direct sum topology. As a consequence, a map $X/A\to Y$ is continuous if and only if its composition with q is a continuous map $X\cup\{*\}\to Y$.

We denote by H the singular homology functor with coefficients in the field $\mathbb{F}$. We consider singular homology theory based on simplices. Recall, that for $d\in\mathbb{N}$ the *standard d-dimensional simplex* is the set $\Delta_d := \{x\in\mathbb{R}^{d+1}\colon\ \sum_i x_i = 1,\ x_i\geq 0\}$. A d-dimensional *singular simplex* on a topological space X is a continuous map $\Delta_d\to X$. A *chain* is a formal linear combination $\sum_i a_i\sigma_i$, where $a_i\in\mathbb{F}$ and σ_i are singular simplices of the same dimension. A chain is called d-dimensional provided all σ_i are d-dimensional. The *support* of a d-dimensional chain $c := \sum_i a_i\sigma_i$, $a_i\neq 0$, is defined as

$$|c| := \bigcup_i \sigma_i(\Delta_d).$$

The *group of chains* on X is denoted by $S(X)$ (actually, in our considerations it is a graded vector space over $\mathbb{F}$). By ∂ we denote the *boundary operator* $S(X)\to S(X)$. A continuous map $f\colon X\to Y$ induces the *chain map* $S(f)\colon S(X)\to S(Y)$. If $A\subset X$ (i.e., (X,A) is a *topological pair*), we treat $S(A)$ as a subspace of $S(X)$ (i.e., if $c \in S(X)$ and $|c| \subset A$, then we regard c also as an element of $S(A)$). By the *group of cycles* on (X,A) we mean

$$Z(X,A) := \{z \in S(X) \colon \partial z \in S(A)\}.$$

The homology class of a cycle $z \in Z(X,A)$ is denoted by $[z]_{(X,A)}$. Recall that $[z]_{(X,A)} = [z']_{(X,A)}$ provided there exist $w \in S(X)$ and $a \in S(A)$ such that

$$z - z' = \partial w + a.$$

The chain map $S(f)$ induced by a continuous map $f \colon (X,A) \to (Y,B)$ maps $Z(X,A)$ to $Z(Y,B)$.

A topological space is called a Euclidean neighborhood retract (shortly: ENR) if it is homeomorphic to a neighborhood retract in a Euclidean space. For properties of ENRs we refer to [6]. In particular,

Proposition 2.3 ([6, Exercise IV.8.13.6])**.** *If (X,A) is a pair of ENRs and A is closed in X then the quotient map induces an isomorphism $H(X,A) \to H(X/A,*)$.*

3. Isolated invariant sets and Conley index

3.1. **Local dynamical systems.** Throughout the rest of the paper we assume that X is a metrizable locally compact space. By a *continuous local dynamical system* (shortly: a *continuous system*) on X we mean a continuous map $\phi \colon D \to X$, where D is an open subset of $X \times \mathbb{R}$, such that for every $x \in X$,

(D1) $D_x := \{t \in \mathbb{R} \colon (x,t) \in D\}$ is an interval containing 0,

(D2) $\phi(x,0) = x$,

(D3) $\forall t \in D_x \colon \{s \in D_{\phi(x,t)} \iff s+t \in D_x,\ \phi(x,t+s) = \phi(\phi(x,t),s)\}$.

Frequently we write $\phi_t(x)$ instead of $\phi(x,t)$ and if $A \subset X$ and $J \subset \mathbb{R}$ we write $\phi(A,J)$ instead of $\phi(A \times J)$. A set of the form $\phi(x,[0,t])$ for some $x \in X$ and $0 \le t < D_x$ is called a *segment*; x is its *starting point* and t is its *length*. (These notions are uniquely determined provided x is not a stationary point and t is less than the minimal period of x in case x is periodic.)

A *discrete local dynamical system* (or shortly: a *discrete system*) on X is a map $f \colon U \to X$ such that U is an open subset of X and f is homeomorphism onto its image $f(U)$.

In the case $D = X \times \mathbb{R}$ the system ϕ is called *global* (or called a *continuous dynamical system*). Similarly, f is called a *global system* if it is a homeomorphism $X \to X$.

Let $S \subset X$. The set S is called *invariant* for a continuous system $\phi \colon D \to X$ (a discrete system $f \colon U \to X$) if for every $x \in S$, $D_x = \mathbb{R}$ and $\phi_t(S) = S$ for $t \in \mathbb{R}$ (respectively if $S \subset U$ and f maps S homeomorphically onto S).

Let $\phi \colon D \to X$ be a continuous system and let $x \in X$ be such that $[0,\infty) \subset D_x$. The *ω-limit set of x* is defined as

$$\omega(x) := \bigcap_{t \ge 0} \mathrm{cl}(\phi(\phi_t(x),[0,\infty))).$$

If $f \colon U \to X$ is a discrete system and $f^n(x) \in U$ for every $n \in \mathbb{N}$, define

$$\omega(x) := \bigcap_{n \in \mathbb{N}} \mathrm{cl}\{f^{n+k}(x) \colon k \in \mathbb{N}\}.$$

In both cases, for each x its ω-limit set is closed and invariant.

3.2. **Isolated invariant sets, index pairs, and index maps.** Let $A \subset X$. Let $\phi\colon D \to X$ be a continuous system and let $f\colon U \to X$ be a discrete one. By the *invariant part* of A (denoted $\operatorname{Inv}(A)$ or, more precisely, $\operatorname{Inv}(A,\phi)$ and $\operatorname{Inv}(A,f)$, respectively) we mean the maximal invariant set contained in A. It is a compact set provided A is compact. An invariant set S is called *isolated* if it is equal to the invariant part of some neighborhood N of S. Such an N is then called an *isolating neighborhood* (for S) provided it is compact.

Let (N,L) be a pair of compact subsets of X. Assume first ϕ is a continuous system on X. The pair (N,L) is called an *index pair for ϕ* if

(P1) $\quad \operatorname{Inv}(\operatorname{cl}(N\setminus L)) \subset \operatorname{int}(N\setminus L)$,

(P2C) $\quad \forall x\in L\colon \{\phi(x,[0,t]) \subset N \Rightarrow \phi(x,[0,t]) \subset L\}$,

(P3C) $\quad \forall x\in N\colon \{\phi(x,D_x\cap[0,\infty)) \not\subset N \Rightarrow \exists t\geq 0\colon \phi(x,[0,t])\subset N, \phi_t(x)\in L\}$.

An index pair (N,L) is called *regular* if the map

$$\sigma\colon N\to[0,\infty],\ \sigma(x) := \begin{cases} \sup\{t>0\colon \phi(x,[0,t])\subset N\setminus L\} & \text{if } x\in N\setminus L,\\ 0 & \text{if } x\in L,\end{cases}$$

called the *exit-time map*, is continuous (compare [27, Definition 5.1]).

Assume now $f\colon U\to X$ is a discrete system on X and $L\subset N\subset U$. The pair (N,L) is called an *index pair for f* if it satisfies (P1) and

(P2D) $$N\cap f(L)\subset L,$$

(P3D) $$N\cap f^{-1}(X\setminus N)\subset L.$$

The pair (N,L) is called a *weak index pair for f* if the conditions (P1) and (P2D) hold, and

(P3W) $$\operatorname{cl}(f(N)\setminus N)\cap N\subset L.$$

Observe that an index pair is also a weak index pair. Define $f_{(N,L)}\colon N/L\to N/L$ as

$$f_{(N,L)}([x]) := \begin{cases} f(x) & \text{if } x, f(x)\in N\setminus L,\\ * & \text{otherwise.}\end{cases}$$

In particular, $f_{(N,L)}(*)=*$.

Proposition 3.1. *If* (P2D) *is satisfied, then $f_{(N,L)}$ is continuous if and only if* (P3W) *holds.* □

It follows by Proposition 3.1 that $f_{(N,L)}$ is continuous provided (N,L) is a weak index pair. In that case $f_{(N,L)}$ is called the *index map*. If S is an isolated invariant set for ϕ (or f) and $S=\operatorname{Inv}(\operatorname{cl}(N\setminus L))$, we call (N,L) an *index pair* for (S,ϕ) (for (S,f), respectively).

Proposition 3.2. *Let $(N,L)\subset(N',L')$ be weak index pairs for (S,f) and*

(2) $$f(N)\cap N'\subset N.$$

Let $i\colon N/L\to N'/L'$ denote the map induced by the inclusion. Then

(a) the diagram

$$\begin{array}{ccc} N/L & \xrightarrow{\quad i\quad} & N'/L' \\ \downarrow{\scriptstyle f_{(N,L)}} & & \downarrow{\scriptstyle f_{(N',L')}} \\ N/L & \xrightarrow{\quad i\quad} & N'/L' \end{array}$$

commutes;

(b) if, moreover,

$$L' \subset N \tag{3}$$

then there exists n_0 such that for every $n \geq n_0$ there exists a continuous map g such that the diagram

$$\begin{tikzcd} N/L \arrow[r, "i"] \arrow[d, "f^n_{(N,L)}"'] & N'/L' \arrow[d, "f^n_{(N',L')}"] \arrow[dl, "g"'] \\ N/L \arrow[r, "i"] & N'/L' \end{tikzcd}$$

commutes.

Proof. Observe first that the quotient topology on $N/(N \cap L') \subset N'/L'$ coincides with the topology induced from N'/L' because N and L' are closed in N'. Let the maps

$$i' \colon N/L \to N/(L' \cap N), \quad i'' \colon N/(L' \cap N) \hookrightarrow N'/L'$$

be induced by the inclusions. Define

$$r \colon N/(L' \cap N) \to N/(L' \cap N),$$
$$r([x]) := \begin{cases} f(x) & \text{if } x \in N \setminus L' \text{ and } f(x) \in N \setminus L', \\ *, & \text{elsewhere.} \end{cases}$$

By (2), r is a restriction of $f_{(N',L')}$, hence it is continuous and the right-hand side rectangle in the diagram below commutes.

$$\begin{tikzcd} N/L \arrow[r, "i'"] \arrow[d, "f_{(N,L)}"'] & N/(L' \cap N) \arrow[r, "i''"] \arrow[d, "r"] & N'/L' \arrow[d, "f_{(N',L')}"] \\ N/L \arrow[r, "i'"] & N/(L' \cap N) \arrow[r, "i''"] & N'/L' \end{tikzcd}$$

The left-hand side rectangle commutes as well, because if $x \in L' \cap N$ and $f(x) \in N \subset N'$, then $f(x) \in L'$. Since $i = i'' \circ i'$, the proof of (a) is finished.

In order to prove (b) note that

$$S = \operatorname{Inv}(\operatorname{cl}(N \setminus L)) = \operatorname{Inv}(\operatorname{cl}(N' \setminus L')).$$

We assert

$$\exists k_0 \in \mathbb{N} \colon \{\{x, f(x), \ldots, f^k(x)\} \subset L' \setminus L \Rightarrow k < k_0\}, \tag{4}$$

$$\exists k_1 \in \mathbb{N} \colon \{\{x, f(x), \ldots, f^k(x)\} \subset N' \setminus N \Rightarrow k < k_1\}. \tag{5}$$

Indeed, if such a k_0 does not exist, there is an x_0 such that

$$\{f^k(x_0) \colon k \in \mathbb{N}\} \subset L' \setminus L,$$

hence

$$\varnothing \neq \omega(x_0) \subset \operatorname{cl}(L' \setminus L) \subset L'.$$

On the other hand, since $\omega(x_0)$ is invariant and $L' \subset N$,

$$\omega(x_0) \subset \operatorname{Inv}(\operatorname{cl}(L' \setminus L)) \subset \operatorname{Inv}(\operatorname{cl}(N \setminus L)) = S \subset \operatorname{int}(N' \setminus L') \subset X \setminus L',$$

a contradiction. Similarly, if there is no such k_1 then there exists an x_1 such that

$$\varnothing \neq \omega(x_1) \subset \operatorname{cl}(N' \setminus N) = N' \setminus \operatorname{int} N.$$

Since by (3),

$$\omega(x_1) \subset \mathrm{Inv}(\mathrm{cl}(N' \setminus N)) \subset \mathrm{Inv}(\mathrm{cl}(N' \setminus L')) = S \subset \mathrm{int}(N \setminus L) \subset \mathrm{int} N,$$

again we get a contradiction. Let k_0 and k_1 satisfy (4) and (5), respectively. Let $x \in N' \setminus L'$. If $f^{k_1}_{(N',L')}([x]) \neq *$ then $f^k(x) \in N \setminus L'$ for some $k = 0, \dots, k_1$ by the choice of k_1 and thus $f^m(x) \in N \setminus L'$ for each $m = k, \dots, k_1$ by (2). In particular, $f^{k_1}(x) \in N \setminus L'$, hence we can define a map

$$p\colon N'/L' \to N/L'$$

as $f^{k_1}_{(N',L')}$ restricted in the codomain. It follows that p is continuous. Since $N \subset N'$,

$$f(L') \cap N \subset L'. \tag{6}$$

Let $y \in L' \setminus L$. Then $y \in N \setminus L$ by (3) and $f^{k_0}_{(N,L)}([y]) = *$ by (6) and the choice of k_0, hence the map

$$q\colon N/L' \to N/L$$

given by

$$q([x]) := \begin{cases} f^{k_0}(x) & \text{if } x \in N \setminus L',\ \{f(x), \dots, f^{k_0}(x)\} \subset N \setminus L, \\ * & \text{elsewhere} \end{cases}$$

is a factorization of $f^{k_0}_{(N,L)}$, hence it is continuous. The choice of p and q implies the upper-right and lower-left triangles in the following diagram

$$\begin{tikzcd}
N/L \arrow[rr, "i"] \arrow[d, "f^{k_1}_{(N,L)}"'] & & N'/L' \arrow[dl, "p"'] \arrow[d, "f^{k_1}_{(N',L')}"] \\
N/L \arrow[r, "i'"] \arrow[d, "f^{k_0}_{(N,L)}"'] & N/L' \arrow[r, "i''"] \arrow[dl, "q"'] & N'/L' \arrow[d, "f^{k_0}_{(N',L')}"] \\
N/L \arrow[rr, "i"] & & N'/L'
\end{tikzcd}$$

commute. Let $[x] \in N/L$. It follows by (2),

$$p(i([x])) = \begin{cases} f^{k_1}(x) & \text{if } \{x, f(x), \dots, f^{k_1}(x)\} \subset N \setminus L', \\ * & \text{elsewhere.} \end{cases} \tag{7}$$

The right-hand side of (7) is equal to $i'(f^{k_1}_{(N/L)}([x]))$ as a consequence of (6), hence we get the commutativity of the upper trapezoid in the diagram. Moreover, if $[x] \in N/L'$ and $q([x]) \in N \setminus L'$ then necessarily $x, f(x), \dots, f^{k_0}(x) \in N \setminus L'$ by (6). It follows

$$i(q([x]) = \begin{cases} f^{k_0}(x) & \text{if } \{x, f(x), \dots, f^{k_0}(x)\} \subset N \setminus L', \\ * & \text{elsewhere.} \end{cases} \tag{8}$$

By (2), the right-hand side of (8) is equal to $f^{k_0}_{(N',L')}(i''([x]))$, hence the lower trapezoid, and therefore the whole diagram, commutes. Put $n_0 = k_0 + k_1$ and $g_0 = q \circ p$. For $n \geq n_0$,

$$g := f^{n-n_0}_{(N,L)} \circ g_0$$

is the required map. $\square$

3.3. **Homology Conley index for discrete systems.** Let R be a fixed normal, consistent retractor. If (N,L) is an index pair for (S,f) then $\mathrm{CH}(S,f)$, the *homology Conley index* of (S,f) (over the retractor R), is defined as the conjugacy class of the automorphism $RH(f_{(N,L)})$. One can prove that every neighborhood of S contains an index pair for (S,f) and the definition of the index is independent of the choice of a weak index pair. In fact, index pairs in the definition of the index can be replaced by weak index pairs.

Proposition 3.3. *If (N,L) is a weak index pair for (S,f), then the conjugacy class of $RH(f_{(N,L)})$ is equal to* $\mathrm{CH}(S,f)$.

Proof. Indeed, let (N,L) be a weak index pair. Put

$$L' := L \cup \mathrm{cl}\{x \in N \colon\, f(x) \notin N\}.$$

Then (N,L') is an index pair and the diagrams in Proposition 3.2 (for $N = N'$) commute with $n = 1$. The result follows by Proposition 2.2. □

3.4. **Isolated invariant sets for a continuous system and its discretizations.** Let ϕ be a continuous system on X and let $h > 0$. The following proposition is proved in [17]. We provide the proof here for the sake of completeness.

Proposition 3.4. *S is an isolated invariant set for ϕ if and only if it is an isolated invariant set for ϕ_h. Moreover, if N is an isolating neighborhood of S for ϕ_h then N is also an isolating neighborhood of S for ϕ.*

Proof. If S is invariant for ϕ then it is obviously invariant for ϕ_h. Moreover, let N be an isolating neighborhood of S with respect to ϕ. Then there exists a compact neighborhood K of S such that $\phi(K,[-h,h]) \subset N$. It follows K is an isolating neighborhood of S with respect to ϕ_h. Conversely, let S be an isolated invariant set for ϕ_h and let N be its isolating neighborhood. Then there exists $\varepsilon > 0$, such that $\phi(S,[-\varepsilon,\varepsilon]) \subset N$. Since the set $\phi(S,[-\varepsilon,\varepsilon])$ is invariant with respect to ϕ_h,

$$\phi(S,[-\varepsilon,\varepsilon]) = S,$$

which implies $\phi(S,\mathbb{R}) = S$. Since each invariant set for ϕ contained in N is an invariant set for ϕ_h, N is an isolating neighborhood of S for ϕ. □

4. Conley index of Poincaré maps

4.1. **Periodic non-autonomous equations.** Let n be a positive integer. For $A \subset \mathbb{R} \times \mathbb{R}^n$ and $t \in \mathbb{R}$ put

$$A_t := \{x \in \mathbb{R}^n \colon\, (t,x) \in A\}.$$

Let Ω be an open subset of $\mathbb{R} \times \mathbb{R}^n$. Assume $f \colon \Omega \to \mathbb{R}^n$ is a time-dependent vector-field such that the equation

$$\dot{x} = f(t,x) \tag{9}$$

has the uniqueness property of the initial-value problem

$$x(t_0) = x_0 \tag{10}$$

associated to (9) (for example, f is smooth). Denote by

$$t \mapsto \Phi_{t_0,t}(x_0)$$

the solution of the problem (9),(10). Let $T > 0$. In the sequel we assume Ω and f are T-periodic with respect to the first variable, i.e.,

$$\Omega_{t+T} = \Omega_t,$$
$$f(t+T,x) = f(t,x)$$

for $t \in \mathbb{R}$ and $(t,x) \in \Omega$. It follows, in particular, that

$$\Phi_{t_0,t} = \Phi_{t_0+T,t+T}.$$

We call

$$P := \Phi_{0,T}$$

the *Poincaré map* for the problem (9),(10). Its domain is equal to the set

$$\{x \in \Omega_0 \colon \Phi_{0,t}(x) \text{ is defined for all } t \in [0,T]\}.$$

Let $[t]$ denote the modulo-T class of $t \in \mathbb{R}$ in $\mathbb{R}/T\mathbb{Z}$ and let Σ denote the quotient space of Ω obtained by the identification of $t \times \Omega_t$ with $(t+T) \times \Omega_t$. Hence,

$$\Sigma = \bigcup_{t\in[0,T)} [t] \times \Omega_t \subset \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n.$$

Thus, the equation (9) induces two continuous systems ϕ on Σ and ψ on Ω given by

$$\phi_t([\tau],x) = ([\tau+t], \Phi_{\tau,\tau+t}(x)),$$
$$\psi_t(\tau,x) = (\tau+t, \Phi_{\tau,\tau+t}(x)).$$

Proposition 4.1. *Let $S \subset \Sigma$. The following conditions are equivalent:*

(i) S is isolated invariant for ϕ,
(ii) S is isolated invariant for ϕ_h, for every $h > 0$,
(iii) S is isolated invariant for ϕ_h, for some $h > 0$,
(iv) S_0 is isolated invariant for the Poincaré map P and $S_t = \Phi_{0,t}(S_0)$ for each $t \in (0,T)$.

Proof. The equivalence of (i), (ii), and (iii) follows by Proposition 3.4. Let (ii) hold and let N be an isolating neighborhood of S in ϕ_T. It follows N_0 is an isolating neighborhood of S_0 with respect to P. Moreover, let $t \in [0,T)$. Then $\Phi_{0,t}(S_0) \subset S_t$ and $\Phi_{t,-t}(S_t) \subset S_0$ by the invariance of S with respect to ϕ (since (ii) implies (i)), hence $\Phi_{0,t}(S_0) = S_t$ and (iv) is satisfied. Finally, assume (iv) holds. For every $t \in (0,T)$, the set S_t is isolated invariant for $\Phi_{t,t+T}$ since that map is conjugated to P. For $t \in [0,T)$ let N_t be an isolating neighborhood of S_t in $\Phi_{t,t+T}$. Actually, for each $t \in [0,T)$ there exists an $\varepsilon_t > 0$ such that N_t is also an isolating neighborhood of S_τ provided $|\tau - t| < \varepsilon_t$. Let $t_1, \dots, t_k \in [0,T)$ be such that

$$[0,T] \subset \bigcup_{i=1}^{k} (t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i}).$$

Then

$$N := \bigcup_{i=1}^{k} \bigcup_{\tau\in(t_i-\varepsilon_{t_i}, t_i+\varepsilon_{t_i})} [\tau] \times N_{t_i}$$

is an isolating neighborhood of S in ϕ_T. Hence, (iii) follows. □

4.2. **Contiguous cycles.** For $B \subset \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n$ define

$$\widetilde{B} := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \colon ([t],x) \in B\}.$$

By abuse of notation, for $t \in \mathbb{R}$ we write B_t instead of $\widetilde{B}_t$.

Let (N,L) be a pair of subsets of Σ. Assume $a,b \in \mathbb{R}$, $a \leq b$. Let $u \in Z(N_a,L_a)$ and $v \in Z(N_b,L_b)$. The pair (u,v) is called a *pair of contiguous cycles over* $[a,b]$ if there exist chains $w \in S(\widetilde{N} \cap ([a,b] \times \mathbb{R}^n))$ and $z \in S(\widetilde{L} \cap ([a,b] \times \mathbb{R}^n))$ such that

$$\partial w = a \times u - b \times v + z,$$

where $a \times u \in Z(a \times N_a, a \times L_a)$ and $b \times v \in Z(b \times N_b, b \times L_b)$ correspond to u and v via the embeddings $x \mapsto (a,x)$ and $x \mapsto (b,x)$, respectively.

Let $h > 0$. A pair of contiguous cycles (u,v) is called *h-movable* provided there exist w and z as above such that

$$\psi(|w|,[0,h]) \subset \widetilde{N}, \quad \psi(|z|,[0,h]) \subset \widetilde{L}.$$

Lemma 4.2. *If (u,v) is a pair of h-movable contiguous cycles over $[a,b]$, then*

$$\psi(|a \times u|,[0,h]) \subset \widetilde{N}, \quad \psi(|b \times v|,[0,h]) \subset \widetilde{N}.$$

Proof.

$$\psi(|a \times u - b \times v|,[0,h]) = \psi(|\partial w - z|,[0,h]) \subset \psi(|w|,[0,h]) \cup \psi(|z|,[0,h]) \subset \widetilde{N}.$$

Since $|a \times u| \cap |b \times v| = \varnothing$, the result follows. □

Lemma 4.3. *If (u,v) is a pair of contiguous cycles over $[a,b]$ and (v,w) is a pair of contiguous cycles over $[b,c]$ then (u,w) is a pair of contiguous cycles over $[a,c]$. Moreover, if (u,v) and (v,w) are h-movable, then (u,w) is also h-movable.* □

Lemma 4.4. *If (u_i,v_i) are pairs of contiguous cycles over $[a,b]$ and $\lambda_i \in \mathbb{F}$, $i = 1,\ldots,k$, then $\left(\sum_{i=1}^k \lambda_i u_i, \sum_{i=1}^k \lambda_i v_i\right)$ is a pair of contiguous cycles over $[a,b]$. Moreover, this pair is h-movable provided all (u_i,v_i) are h-movable.* □

4.3. **The main theorem.** Let $h > 0$ and let (N,L) be a weak index pair for ϕ_h. Put

$$S := \operatorname{Inv}(\operatorname{cl}(N \setminus L), \phi_h).$$

By Proposition 4.1, S_0 is an isolated invariant set for the Poincaré map P. The main theoretical result of the present paper is

Theorem 4.5. *Let $T/h \in \mathbb{Q}$. Assume N_0 and L_0 are ENR's,*

$$k := \dim H(N_0,L_0),$$

$A = [a_{ij}]$ is a graded $(k \times k)$-matrix over $\mathbb{F}$, and

$$\left(u_j, \sum_{i=1}^k a_{ij} u_i\right), \ j = 1,\ldots,k,$$

are h-movable pairs of contiguous cycles over $[0,T]$ such that $\{[u_j]_{(N_0,L_0)} \colon j = 1,\ldots,k\}$ is a basis of $H(N_0,L_0)$. Then $\operatorname{CH}(S_0,P)$ is equal to the conjugacy class of RA.

A proof is postponed to the next section. An example application of Theorem 4.5 is the following corollary, which follows from [30, Lemma 5.2] (see also [16]).

Corollary 4.6. *Under the assumptions of Theorem* 4.5*, if*

$$\Lambda(A) \neq 0,$$

then the equation (9) *has a T-periodic solution.*

In Theorem 4.5, the assumption on h-movability is essential. Indeed, consider the equation

$$\dot{x} = 1$$

on $\mathbb{R}$. We treat it as T-periodic for some $T > 0$. Let $h = 2$ and for each $t \in \mathbb{R}$ let

$$N_t := [0,1] \cup [2,6],\ L_t := [4,6].$$

It follows that (N,L) is an index pair for ϕ_h. Obviously, the invariant part of N is empty, hence the Conley index of the Poincaré map is trivial. On the other hand, let u be equal to the singular 0-dimensional simplex 1 in the interval $[0,1]$. Then u is a cycle which is equal to its homology class and it is a generator of $H(N_0,L_0) \cong \mathbb{F}$. Moreover, (u,u) is a pair of contiguous cycles. Indeed, the corresponding chains w and z can be given as w equal to the 1-dimensional singular simplex $[0,T] \times 1$ in $[0,T] \times [0,1]$ and $z = 0$. Note however, the pair (u,u) is not h-movable. Since $A = (1)$, its conjugacy class is nontrivial.

5. Proof of Theorem 4.5

5.1. **Reduction to global systems.** Let $g\colon \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n \to [0,1]$ be a smooth function such that for some compact set $K \subset \Sigma$ satisfying $N \subset \operatorname{int} K$,

$$\mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n \setminus \operatorname{int} K \subset g^{-1}(0),\ N \subset g^{-1}(1),$$

Put $\widetilde{g}(t,x) := g([t],x)$ for $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. Replacing (9) by

$$\dot{x} = \widetilde{g}(t,x) f(t,x) \tag{11}$$

we get global systems ϕ' on $\mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n$ and ψ' on $\mathbb{R} \times \mathbb{R}^n$ corresponding to ϕ and ψ, respectively. Moreover, the set S remains isolated for ϕ', (N,L) is an index pair for (S,ϕ'), the h-movability property (c.f. Subsection 4.2) is preserved if ψ is replaced by ψ'. It is also clear that

$$\mathrm{CH}(S_0,P') = \mathrm{CH}(S_0,P),$$

where P' denotes the Poincaré map associated to (11). Hence, in the sequel we are able to assume

$$\Sigma = \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n,$$
$$\Omega = \mathbb{R} \times \mathbb{R}^n,$$

and the equation (9) generates global systems ϕ and ψ.

5.2. **Auxiliary index pairs.** In order to find the Conley index of (S_0,P) one should find a weak index pair for it. To this aim we define several auxiliary sets. Put

$$N' := \bigcup\{\sigma \subset N\colon \sigma \text{ is a segment of length } h\},$$
$$L' := \bigcup\{\tau \subset N\colon \tau \text{ is a segment starting at a point in } L\},$$
$$N^* := N' \cup L'.$$

It follows immediately that

$$L \subset L' \subset N^* \subset N.$$

Lemma 5.1. *The sets N', L' and N^* are compact.*

Proof. We only show compactness of L' here, the proof for N' is similar. Using that L is positive invariant under ϕ_h, one can see that

$$L' = \bigcup\{\tau \subset N\colon\ \tau \text{ is a segment with length in } [0,h] \text{ starting at a point in } L\ \}.$$

Let $\{x_n\}$ be a sequence in L'. Then there are $y_n \in L$ and $t_n \in [0,h]$ such that $\phi(y_n,t_n) = x_n$ and $\phi(y_n,[0,t_n]) \subset N$. We choose an increasing sequence of natural numbers $\{n(k) \mid k \in \mathbb{N}\}$ such that $y_{n(k)} \to y_* \in L$ and $t_{n(k)} \to t_* \in [0,h]$. Since $\phi(y_n,[0,t_n]) \subset N$, also $\phi(y_*,[0,t_*]) \subset N$ and therefore $x_{n(k)} \to \phi(y_*,t_*) \in L'$ for $k \to \infty$. □

Lemma 5.2. *If $x \in N'$ and $\phi(x,[0,\varepsilon)) \not\subset N$ for every $\varepsilon > 0$ then $x \in L$.*

Proof. There is a point $y \in N$ such that $\phi(y,[0,h]) \subset N$ and $x = \phi_h(y)$. Then $\phi_h(\phi_{\varepsilon_n}(y)) \notin N$ for some sequence $\varepsilon_n > 0$, $\varepsilon_n \to 0$. By (P3W), $x \in L$. □

Lemma 5.3.

(a) *If $x \in N'$, $\phi_h(x) \in N$ and $\phi(x,[0,h]) \not\subset N$, then $\phi_h(x) \in L'$.*
(b) *If $x \in L'$ and $\phi_h(x) \in N$, then $\phi_h(x) \in L'$.*

Proof. In order to prove (a), let $x \in N'$ and $\phi_h(x) \in N$. By definition of N', there are $t \in [0,h]$ and $y \in N$ such that $\phi(y,[0,h]) \subset N$ and $x = \phi_t(y)$. It follows

$$\phi(x,[0,h-t]) \subset N.$$

There is a number s such that

$$0 < s \le h, \quad \phi(x,[s,h]) \subset N, \quad \exists 0 < \varepsilon_n \to 0\colon\ \phi_{s-\varepsilon_n}(x) \notin N. \tag{12}$$

Then

$$h < s+t \le 2h,$$

since if $\phi(x,[h-t,h]) \subset N$, then $\phi(x,[0,h]) \subset N$, which contradicts the assumption. It follows $\phi_{s+t-\varepsilon_n-h}(y) \in N$, hence (P3W) and (12) imply $\phi_s(x) \in L$ and therefore $\phi_h(x) \in L'$.

For the proof of (b), let $x \in L'$ and $\phi_h(x) \in N$. Let $x = \phi_t(y)$, $y \in L$, and $\phi(y,[0,t]) \subset N$. By (P2D), one can assume $t \in (0,h)$. If $\phi(x,[0,h]) \subset N$ then $\phi(y,[0,t+h]) \subset N$ and $\phi_h(x) \in L'$. If not, there is an s satisfying (12). Assume first

$$h < s+t.$$

Then $\phi_{s+t-h-\varepsilon_n}(y) \in \phi(y,(0,t]) \subset N$, hence $\phi_s(x) \in L$ and $\phi_h(x) \in L'$. Finally, assume

$$s+t \le h.$$

Then $\phi_h(y) \in \phi(x,[s,h]) \subset N$, which implies $\phi_h(y) \in L$ by (P2D). Thus

$$\phi_h(x,[h-t,h]) \subset L'$$

and (b) follows. □

Lemma 5.4. *(N^*,L) is a weak index pair for (S,ϕ_h). Moreover,*

$$\phi_h(N^*) \cap N \subset N^*. \tag{13}$$

Proof. At first we prove (13). Since $\phi(x,[0,h]) \subset N$ implies trivially $x, \phi_h(x) \in N'$, the claim follows immediately by Lemma 5.3. In order to prove (P1) choose U, a neighborhood of S such that $\phi(U,[0,h]) \subset \operatorname{int}(N \setminus L)$. It follows $U \subset N^*$, hence $S \subset \operatorname{int}(N^* \setminus L)$. Therefore

$$S \subset \operatorname{Inv}(\operatorname{cl}(N^* \setminus L)) \subset \operatorname{Inv}(\operatorname{cl}(N \setminus L)) = S,$$

hence (P1) is proved. Since $N^* \subset N$ and (N,L) is a weak index pair, the condition (P2D) is obvious. For a proof of (P3W) assume $x_n \in N^*$, $x_n \to x_0$, $\phi_h(x_0) \in N^*$, and $\phi_h(x_n) \notin N^*$.

It follows by (13), $\phi_h(x_n) \notin N$. Then $\phi_h(x_0) \in L$ because (N,L) is a weak index pair, hence the proof is finished. □

Lemma 5.5. *(N^*, L') is an index pair for (S, ϕ).*

Proof. By Proposition 3.4 and Lemma 5.4, $\mathrm{cl}(N^* \setminus L)$ is an isolating neighborhood of S for ϕ. Obviously $S \cap L' = \varnothing$, hence (P1). The condition (P2C) follows directly by the definition of L'. In order to prove (P3C), let $0 \le t < \infty$ be such that

$$\phi(x,[0,t]) \subset N^*, \quad \phi(x,[0,t+\varepsilon]) \not\subset N^*$$

for every $\varepsilon > 0$. One can assume $x \notin L'$, hence $x \in N'$. It follows $\phi_t(x) \in N'$ and thus $\phi_t(x) \in L$ by Lemma 5.2. □

Lemma 5.6 ([27, Lemma 5.3])**.** *There exists a continuous function $\alpha\colon N^* \to [0,1]$ such that*

$$\begin{aligned}
&\alpha(x) = 1 \Longleftrightarrow \phi(x,[0,\infty)) \subset N^* \text{ and } \omega(x,\phi) \subset S,\\
&\alpha(x) = 0 \Longleftrightarrow x \in L',\\
&t > 0,\ 0 < \alpha(x) < 1,\ \phi(x,[0,t]) \subset N^* \Rightarrow \alpha(\phi_t(x)) < \alpha(x). \quad \square
\end{aligned}$$

Let α be given in Lemma 5.6 and let $0 < c < 1$ be such that

$$\alpha(x) \ge c \ \Rightarrow\ \phi(x,[0,\max\{h,T\}]) \subset N^*. \tag{14}$$

Put

$$L^* := \alpha^{-1}([0,c]).$$

Lemma 5.7 ([27, Remark 5.4])**.** *(N^*, L^*) is a regular index pair for (S,ϕ).* □

Lemma 5.8. *(N^*, L^*) is an index pair for (S, ϕ_h).*

Proof. The condition (P1) is obvious and (P3D) follows by (14). In order to prove (P2D) let $x \in L^*$ and $\phi_h(x) \in N^*$. If $\phi(x,[0,h]) \subset N$, then $\phi(x,[0,h]) \subset N' \subset N^*$ and $\alpha(\phi_h(x)) \le \alpha(x) \le c$, which implies the result. On the other hand, if $\phi(x,[0,h]) \not\subset N$, then $\phi_h(x) \in L' \subset L^*$ follows directly from Lemma 5.3. □

Define $\rho\colon \widetilde{N}^* \cap ([0,T] \times \mathbb{R}^n) \to N_0^*/L_0^*$ by the formula

$$\rho(t,x) := \begin{cases} \Phi_{t,T}(x) & \text{if } \Phi_{t,s}(x) \in N_s^* \setminus L_s^* \text{ for every } s \in [t,T],\\ * & \text{otherwise.}\end{cases}$$

Lemma 5.9. *The map ρ is continuous.*

Proof. Put $\sigma^*\colon \widetilde{N}^* \cap ([0,T] \times \mathbb{R}^n) \to [0,\infty]$,

$$\sigma^*(t,x) := \begin{cases} \sup\{\tau \in (0,\infty)\colon \Phi_{t,s}(x) \in N_s^* \setminus L_s^*\ \forall s \in [t,t+\tau]\} & \text{if } x \in N_t^* \setminus L_t^*,\\ 0 & \text{otherwise.}\end{cases}$$

The map σ^* is continuous by Lemma 5.7. Hence, in particular, if $\sigma(t,x) = T - t$, then $\Phi_{t,T}(x) \in L_T^*$. Since

$$\rho(t,x) = \begin{cases} \Phi_{t,T}(x) & \text{if } \sigma^*(t,x) > T-t,\\ * & \text{otherwise,}\end{cases}$$

the result follows. □

Put

$$N^{**} := \alpha^{-1}([c,1]), \quad L^{**} := \alpha^{-1}(c).$$

Lemma 5.10. *(N^{**}, L^{**}) is a weak index pair for (S, ϕ_τ) for every $\tau \in (0, \max\{h, T\}]$.*

Proof. By the properties of the function α, the set S is maximal invariant with respect to ϕ_τ in N^{**}, hence (P1) follows. By (14), if $x \in L^{**}$, then $\alpha(\phi_\tau(x)) < c$, hence $\phi_\tau(x) \notin N^{**}$ and (P2D) follows. Finally, if $x_n \in N^{**}$, $\phi_\tau(x_n) \notin N^{**}$, $x_n \to x_0$ and $\phi_\tau(x_0) \in N^{**}$, then $\alpha(\phi_\tau(x_0)) = c$ and (P3W) follows. □

By the definition of quotient space,

$$N^{**}/L^{**} = N^*/L^*. \tag{15}$$

It follows, in particular,

$$(\phi_h)_{(N^*,L^*)} = (\phi_h)_{(N^{**},L^{**})}. \tag{16}$$

Also by (15) and Lemma 5.10 we can formulate the following result.

Lemma 5.11. *If $x \in N^*$ and $0 < \tau \leq \max\{h, T\}$, then*

$$(\phi_\tau)_{(N^{**},L^{**})}([x]) = \begin{cases} \phi_\tau(x) & \text{if } \phi_t(x) \in N^* \setminus L^* \text{ for every } t \in [0,\tau], \\ * & \text{elsewhere.} \end{cases}$$

Proof. Let $x \in N^* \setminus L^*$. By (14),

$$\phi_t(x) \in N^*$$

for each $t \in [0, \tau]$. If $\phi_t(x) \notin N^* \setminus L^*$ for some $t \in (0, \tau)$, then there exists $0 < s < t$ such that

$$\alpha(\phi_s(x)) = c,$$

hence $\alpha(\phi_\tau(x)) < c$ and $(\phi_\tau)_{(N^{**},L^{**})}([x]) = *$. □

Corollary 5.12. *(N_0^{**}, L_0^{**}) is a weak index pair for (S_0, P) and for $x \in N_0^*$,*

$$P_{(N_0^{**},L_0^{**})}([x]) = \begin{cases} P(x) & \text{if } \Phi_{0,t}(x) \in N_t^* \setminus L_t^* \text{ for every } t \in [0,T], \\ * & \text{elsewhere.} \end{cases} \qquad \square$$

By Proposition 3.3 and Corollary 5.12, in order to prove Theorem 4.5 one should verify the equality of the conjugacy classes of RA and $RH(P_{(N_0^{**},L_0^{**})})$.

5.3. **Endomorphisms induced by the matrix** A. Let

$$\pi\colon (N_0, L_0) \to (N_0/L_0, *), \quad \pi^*\colon (N_0^*, L_0) \to (N_0^*/L_0, *)$$

be the quotient maps and let

$$I\colon (N_0^*, L_0) \hookrightarrow (N_0, L_0), \quad \iota\colon N_0^*/L_0 \hookrightarrow N_0/L_0$$

be the inclusion maps. Since (N_0, L_0) is a pair of compact ENRs, Proposition 2.3 implies that $H(\pi)$ is an isomorphism. By Lemma 4.2, $u_i \in Z(N_0^*, L_0)$ for each i. Since

$$H(I)[u_i]_{(N_0^*,L_0)} = [u_i]_{(N_0,L_0)},$$

by the commutativity of the diagram

$$\begin{tikzcd} H(N_0^*, L_0) \arrow[r, "H(I)"] \arrow[d, "H(\pi^*)"'] & H(N_0, L_0) \arrow[d, "H(\pi)", "\cong"'] \\ H(N_0^*/L_0, *) \arrow[r, "H(\iota)"] & H(N_0/L_0, *), \end{tikzcd}$$

the vectors

$$\overline{u}_i := H(\pi^*)[u_i]_{(N_0^*,L_0)}$$

are linearly independent in $H(N_0^*/L_0,*)$. Put

$$V := \mathrm{Lin}\{\overline{u}_i \colon i = 1,\dots,k\} \subset H(N_0^*/L_0,*)$$

and denote

$$\overline{u} := (\overline{u}_1,\dots,\overline{u}_k).$$

It is an ordered basis of V. By definition of $A_{\overline{u}}$, it is represented by the matrix A. Considering A as an endomorphism with the same name (by left multiplication), we have the conjugacy

$$A \cong A_{\overline{u}}. \tag{17}$$

Let

$$\kappa\colon N_0^*/L_0 \to N_0^*/L_0^*$$

be induced by the inclusion $(N_0^*,L_0) \subset (N_0^*,L_0^*)$.

Lemma 5.13. *The diagram*

$$\begin{array}{ccc} V & \xrightarrow{H(\kappa)|_V} & H(N_0^*/L_0^*,*) \\ {\scriptstyle A_{\overline{u}}}\big\downarrow & & \big\downarrow{\scriptstyle H(P_{(N_0^{**},L_0^{**})})} \\ V & \xrightarrow{H(\kappa)|_V} & H(N_0^*/L_0^*,*) \end{array}$$

commutes.

Proof. Fix $j = 1,\dots,k$. Let w_j and z_j be chains from the definition of h-movability of the contiguous pair of cycles $\big(u_j, \sum_{i=1}^k a_{ij}u_i\big)$ over $[0,T]$. It follows, in particular,

$$w_j \in S(\widetilde{N}^* \cap [0,T] \times \mathbb{R}^n).$$

Let ρ be given in Lemma 5.9. Then $S(\rho)z_j \in S(*)$ and for every $x \in N_0^*$,

$$\begin{aligned} \rho(T,x) &= \kappa \circ \pi^*(x), \\ \rho(0,x) &= P_{(N_0^{*,*},L_0^{**})}(\kappa \circ \pi^*(x)) \end{aligned}$$

by Corollary 5.12. Note that $\overline{u}_j = [S(\pi^*)u_j] \in V$. The computation on singular chains

$$S(P_{(N_0^{*,*},L_0^{**})})S(\kappa)S(\pi^*)u_j - \sum_i a_{ij}S(\kappa)S(\pi^*)u_i = S(\rho)(0 \times u_j) - S(\rho)\left(T \times \sum_i a_{ij}u_i\right) = \\ \partial S(\rho)w_j - S(\rho)z_j \in \partial S(N_0^*/L_0^*) + S(*)$$

implies the result. □

5.4. **Endomorphisms induced by index maps.** In the sequel we use the following notation. Let $(\overline{N},\overline{L})$ be a weak index pair for (S,ϕ_h). For $t \in \mathbb{R}$ define

$$\Psi_{t,t+h}^{(\overline{N},\overline{L})} \colon \overline{N}_t/\overline{L}_t \to \overline{N}_{t+h}/\overline{L}_{t+h}$$

induced by the restrictions of the corresponding index map, i.e., for $x \in \overline{N}_t$,

$$\Psi_{t,t+h}^{(\overline{N},\overline{L})}([x]) := \begin{cases} \Phi_{t,t+h}(x) & \text{if } x \in \overline{N}_t \setminus \overline{L}_t,\ \ \Phi_{t,t+h}(x) \in \overline{N}_{t+h} \setminus \overline{L}_{t+h}, \\ * & \text{elsewhere.} \end{cases}$$

The map $\Psi^{(\overline{N},\overline{L})}_{t,t+h}$ is continuous. Since T/h is rational,

$$ph = qT \tag{18}$$

for some positive integers p and q. Define

$$\Psi_{(\overline{N},\overline{L})} := \Psi^{(\overline{N},\overline{L})}_{(p-1)h,ph} \circ \ldots \circ \Psi^{(\overline{N},\overline{L})}_{0,h} \colon \overline{N}_0/\overline{L}_0 \to \overline{N}_0/\overline{L}_0.$$

Recall that (N^*,L) and (N^*,L^*) are weak index pairs for (S,ϕ_h) by Lemmas 5.4 and 5.8. As a consequence of (16) and Lemma 5.11 we get

$$\Psi^{(N^*,L^*)}_{t,t+h}([x]) = \begin{cases} \Phi_{t,t+h}(x) & \text{if } \Phi_{t,t+s}(x) \in N^*_{t+s} \setminus L^*_{t+s} \text{ for every } s \in [0,h], \\ * & \text{elsewhere} \end{cases} \tag{19}$$

and by Proposition 3.2 the diagram

$$\begin{array}{ccc} N^*_0/L_0 & \xrightarrow{\ \kappa\ } & N^*_0/L^*_0 \\ \Big\downarrow{\scriptstyle \Psi_{(N^*,L)}} & & \Big\downarrow{\scriptstyle \Psi_{(N^*,L^*)}} \\ N^*_0/L_0 & \xrightarrow{\ \kappa\ } & N^*_0/L^*_0 \end{array} \tag{20}$$

commutes.

Lemma 5.14. *For every $v \in V$,*

$$H(\Psi_{(N^*,L)})v = A^q_{\overline{u}}v.$$

Proof. Denote the elements of A^q by a^q_{ij}; i.e., $A^q = [a^q_{ij}]_{i,j=1,\ldots,k}$. For $j = 1,\ldots,k$ put

$$v_j := \sum_{i=1}^{k} a^q_{ij} u_i.$$

By Lemmas 4.3 and 4.4, each (u_j, v_j) is an h-movable pair of contiguous cycles over $[0,qT]$. Fix $j = 1,\ldots,k$. Let w_j and z_j be chains corresponding to (u_j,v_j) in the definition of h-movability, hence

$$\begin{aligned} &\partial w_j = 0 \times u_j - T \times v_j + z_j, \\ &\forall (t,x) \in |w_j|\ \forall s \in [0,h] \colon \Phi_{t,t+s}(x) \in N^*_{t+s}, \\ &\forall (t,x) \in |z_j|\ \forall s \in [0,h] \colon \Phi_{t,t+s}(x) \in L_{t+s}. \end{aligned}$$

Define a map

$$Q \colon |w_j| \to N^*_0/L_0,$$

as follows. Let $(t,x) \in |w_j|$. By (18) we can assume $t = rh - s$ for some $r = 1,\ldots,p$ and $s \in [0,h]$. Put

$$Q(t,x) := \begin{cases} \Psi^{(N^*,L)}_{(p-1)h,ph} \circ \ldots \circ \Psi^{(N^*,L)}_{rh,(r+1)h}([\Phi_{rh-s,rh}(x)]) & \text{if } r = 1,\ldots,p-1, \\ [\Phi_{ph-s,ph}(x)] & \text{if } r = p. \end{cases}$$

The map Q is well-defined. Indeed, $\Phi_{rh-s,rh}(x) \in N^*_{rh}$ by h-movability, and if $s = h$, then

$$\Psi^{(N^*,L)}_{(r-1)h,rh}([x]) = [\Phi_{(r-1)h,rh}(x)]$$

by definition. Q is continuous since its restriction to each set $|w_j| \cap ([(r-1)h, rh] \times \mathbb{R}^n)$ is continuous. By definition,

$$Q(0,x) = \Psi_{(N^*,L)}(\pi^*(x)),$$
$$Q(qT,x) = \pi^*(x).$$

Moreover, by h-movability,

$$Q(|z_j|) = *.$$

It follows that

$$S(\Psi^{(N^*,L)})S(\pi^*)u_j - \sum_i a^q_{ij} S(\pi^*)u_i = S(Q)(0 \times u_j) - S(Q)(qT \times v_j) =$$
$$\partial S(Q)w_j + S(Q)z_j \in \partial S(N_0^*/L_0) + S(*),$$

which implies the result. $\square$

Lemma 5.15.

$$\Psi_{(N^*,L^*)} = P^q_{(N_0^{**},L_0^{**})}.$$

Proof. Let $x \in N_0^*$. By (19),

$$\Psi_{(N^*,L^*)}([x]) = \begin{cases} \Phi_{0,ph}(x) & \text{if } \Phi_{0,t}(x) \in N_t^* \setminus L_t^* \text{ for every } t \in [0,qT], \\ * & \text{elsewhere,} \end{cases}$$

but the right-hand side is equal to $P^q_{(N^{**},L^{**})}([x])$ by (18) and Corollary 5.12. $\square$

By Proposition 3.2 and (13), there exist $k, k^* \in \mathbb{N}$ and continuous maps g and g^* such that the following diagrams

$$\begin{tikzcd} N^*/L \arrow[r] \arrow[d, "(\phi_h)^{kp}_{(N^*,L)}"'] & N/L \arrow[d, "(\phi_h)^{kp}_{(N,L)}"] \arrow[dl, "g"'] \\ N^*/L \arrow[r] & N/L, \end{tikzcd} \qquad \begin{tikzcd} N^*/L \arrow[r] \arrow[d, "(\phi_h)^{k^*p}_{(N^*,L)}"'] & N^*/L^* \arrow[d, "(\phi_h)^{k^*p}_{(N^*,L^*)}"] \arrow[dl, "g^*"'] \\ N^*/L \arrow[r] & N^*/L^*, \end{tikzcd}$$

in which the horizontal arrows are induced by inclusions, commute. It follows by commutativity that the restrictions of g and g^* to the fiber over zero in $\mathbb{R}/T\mathbb{Z}$ induce continuous maps

$$g_0 \colon N_0/L_0 \to N_0^*/L_0, \quad g_0^* \colon N_0^*/L_0^* \to N_0^*/L_0$$

such that the diagrams

(21)
$$\begin{tikzcd} N_0^*/L_0 \arrow[r, "\iota"] \arrow[d, "\Psi^k_{(N^*,L)}"'] & N_0/L_0 \arrow[d, "\Psi^k_{(N,L)}"] \arrow[dl, "g_0"'] \\ N_0^*/L_0 \arrow[r, "\iota"] & N_0/L_0, \end{tikzcd}$$

(22)
$$\begin{tikzcd} N_0^*/L_0 \arrow[r, "\kappa"] \arrow[d, "\Psi^{k^*}_{(N^*,L)}"'] & N_0^*/L_0^* \arrow[d, "\Psi^{k^*}_{(N^*,L^*)}"] \arrow[dl, "g_0^*"'] \\ N_0^*/L_0 \arrow[r, "\kappa"] & N_0^*/L_0^* \end{tikzcd}$$

commute.

Lemma 5.16.
$$H(\Psi^k_{(N^*,L)})(H(N_0^*/L_0,*)) \subset V.$$

Proof. Let $w \in H(N_0^*/L_0,*)$. Since $H(\iota)|_V$ is an isomorphism, there exists $v \in V$ such that
$$H(\iota)w = H(\iota)v.$$
The above equation, the commutativity of (21), and Lemma 5.14 imply
$$H(\Psi^k_{(N^*,L)})w = H(\Psi^k_{(N^*,L)})v = A^{kq}_{\overline{u}} v \in V,$$
hence the result. □

By Lemma 5.16 define
$$\Gamma\colon H(N_0^*/L_0^*,*) \ni w \mapsto H(\Psi^k_{(N^*,L)})H(g_0^*)w \in V.$$

Lemma 5.17. *The diagram*
$$\begin{tikzcd}
V \arrow[r, "H(\kappa)|_V"] \arrow[d, "A^{(k+k^*)q}_{\overline{u}}"'] & H(N_0^*/L_0^*,*) \arrow[d, "H(P_{(N_0^{**},L_0^{**})})^{(k+k^*)q}"] \arrow[dl, "\Gamma"'] \\
V \arrow[r, "H(\kappa)|_V"] & H(N_0^*/L_0^*,*)
\end{tikzcd}$$
commutes.

Proof. The commutativity of (20) and (22), and Lemma 5.16 imply the diagram
$$\begin{tikzcd}
V \arrow[r, "H(\kappa)|_V"] \arrow[d, "H(\Psi_{(N^*,L)})^{k^*}|_V"'] & H(N_0^*/L_0^*,*) \arrow[d, "H(\Psi_{(N^*,L^*)})^{k^*}"] \arrow[dl, "H(g_0^*)"'] \\
H(N_0^*/L_0,*) \arrow[r, "H(\kappa)"] \arrow[d, "H(\Psi_{(N^*,L)})^k"'] & H(N_0^*/L_0^*,*) \arrow[d, "H(\Psi_{(N^*,L^*)})^k"] \\
V \arrow[r, "H(\kappa)|_V"] & H(N_0^*/L_0^*,*)
\end{tikzcd}$$
commutes. The conclusion is a consequence of Lemmas 5.14 and 5.15. □

5.5. **Final step of the proof.** By Proposition 2.2, the equation (17), and Lemmas 5.13 and 5.17,
$$RA \cong RA_{\overline{u}} \cong RH(P_{(N_0^{**},L_0^{**})}).$$
The conjugacy class of the latter is equal to $\mathrm{CH}(S_0,P)$ by Corollary 5.12, hence Theorem 4.5 is proved.

6. Rigorous numerics of dynamical systems

In this section and the following one, we present the algorithm used to check the prerequisites of Theorem 4.5. The algorithm is based on the rigorous construction of a weak index pair (N,L) of ϕ_h as in [20]. The index pair is a pair of cubical sets [11]. To facilitate efficient computability of homology groups, in the algorithms we replace singular homology by cubical homology as defined in [11]. We still can use Theorem 4.5 because singular and cubical homology are isomorphic on cubical sets.

The algorithm constructs 1-chains z, v and a 2-chain w as linear combinations of elementary cubes. In the course of their construction, the algorithm has to ensure the movability condition. We first describe how we use cubes to combinatorially model a dynamical system. For more details about these methods, we refer to [11].

6.1. **Discretizing the space.** We use the following unit cubes filling $\mathbb{R}^3$ as a combinatorial model for the quotient space $\Sigma = \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^2$.

Definition 6.1. An *elementary interval* I is an interval of the form $I = [i] := [i,i]$ or $I = [i,i+1]$ for an $i \in \mathbb{Z}$. An *elementary cube* is a product of intervals $Q = I_1 \times I_2 \times I_3 \subset \mathbb{R}^3$, where each I_i is an elementary interval. The *dimension* of Q is the number of intervals of the form $[i,i+1]$ in this product. The word *cube* always refers to an elementary cube in this article. When $\mathcal{A}$ is a set of elementary cubes, let $|\mathcal{A}| = \bigcup_{Q\in\mathcal{A}} Q \subset \mathbb{R}^3$. If $\mathcal{A}$ is finite, we call $|\mathcal{A}|$ a *cubical set*.

A k-dimensional elementary cube is called a *k-cube*. We often use special names depending on dimension: a *vertex* is a 0-cube, an *edge* is a 1-cube, a *square* is a 2-cube, and a *full cube* is a 3-cube. For a set $\mathcal{A}$ of cubes, let $\mathcal{A}^k := \{Q \in \mathcal{A} \mid \dim Q = k\}$.

Given a subdivision parameter $m \geq 1$, the extended phase space $\Sigma = \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^2$ is covered as follows. Let

$$\mathcal{X} = \{[i,i+1] \times [j,j+1] \times [k,k+1] \mid i \in \{0,1,\dots,2^m-1\}, j,k \in \mathbb{Z}\}.$$

To cover a bounded region, we choose $a,b > 0$ and define

$$\begin{aligned} &\alpha\colon |\mathcal{X}| \to \Omega = \mathbb{R} \times \mathbb{R}^2, \\ &(x_1,x_2,x_3) \mapsto \left(T \cdot \frac{x_1}{2^m},\ a\left(\frac{x_2}{2^{m-1}} - 1\right),\ b\left(\frac{x_3}{2^{m-1}} - 1\right)\right). \end{aligned}$$

Let $q\colon \Omega \to \Sigma$ be the quotient map, and let $p = q \circ \alpha$. Note that $\alpha([0,2^m]^3) = [0,T] \times [-a,a] \times [-b,b]$ and $p(|\mathcal{X}|) = \Sigma$.

Definition 6.2. For a set $\mathcal{A}$ of cubes, let

$$[\![\mathcal{A}]\!] := p(|\mathcal{A}|) = \bigcup_{Q\in\mathcal{A}} p(Q) \subset \Sigma$$

be its *geometric realization*.

Observe that $[\![\mathcal{A} \cup \mathcal{B}]\!] = [\![\mathcal{A}]\!] \cup [\![\mathcal{B}]\!]$ and $[\![\mathcal{A} \cap \mathcal{B}]\!] \subset [\![\mathcal{A}]\!] \cap [\![\mathcal{B}]\!]$ for any sets $\mathcal{A}, \mathcal{B}$ of cubes.

6.2. **Discretizing the generator of the dynamical system.**

Definition 6.3. For a set $\mathcal{A}$ of cubes, a *multivalued combinatorial map* $\mathcal{F}$ on $\mathcal{A}$ is a map from $\mathcal{A}$ to its power set. This is written as $\mathcal{F}\colon \mathcal{A} \rightrightarrows \mathcal{A}$.

Definition 6.4. Let $I \subset \mathbb{Z}$ be an interval of integers containing 0. For a combinatorial multivalued map $\mathcal{F}\colon \mathcal{A} \rightrightarrows \mathcal{A}$, a *solution* through $Q \in \mathcal{A}$ is a map $\Gamma\colon I \to \mathcal{A}$ such that

(i) $\Gamma(0) = Q$, and
(ii) $\Gamma(k+1) \in \mathcal{F}(\Gamma(k))$ for all k such that $k, k+1 \in I$.

Definition 6.5. For a set $\mathcal{M} \subset \mathcal{A}$, let

(i) $\operatorname{Inv}(\mathcal{M},\mathcal{F}) = \{Q \in \mathcal{M} \mid \text{there is a solution } \mathbb{Z} \to \mathcal{M} \text{ through } Q\}$,
(ii) $\operatorname{Inv}^{\pm}(\mathcal{M},\mathcal{F}) = \{Q \in \mathcal{M} \mid \text{there is a solution } \mathbb{Z}^{\pm} \to \mathcal{M} \text{ through } Q\}$.

Definition 6.6. Given a continuous map $f\colon \Sigma \to \Sigma$, the map $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is a *combinatorial enclosure* of f if for every $Q \in \mathcal{X} : f(p(Q)) \subset \operatorname{int}[\![\mathcal{F}(Q)]\!]$.

We are interested in the numerical study of differential equations of the form

$$\dot{z} = f(t,z)$$

with $z \in \mathbb{C} \cong \mathbb{R}^2$ and $f(t,z) = f(t+T,z)$ for every $t \in \mathbb{R}$. This yields flows ψ on $\Omega = \mathbb{R} \times \mathbb{R}^2$ and ϕ on $\Sigma = \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^2$ as described in Section 4. We assume that the flows are defined for all times $t \in \mathbb{R}$ using the argument from Subsection 5.1. The time-h maps ψ_h on $\Omega = \mathbb{R} \times \mathbb{R}^2$ and ϕ_h on $\Sigma = \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^2$ describe discrete dynamical systems.

Given a product $V \subset \Omega$ of closed intervals (an *interval vector*) and an interval $J \subset [0,\infty)$, we use the software library CAPD [36] to find an interval vector $V' \subset \Omega$ such that $\psi(V,J) \subset \operatorname{int}(V')$. We use higher-order Taylor methods, but the overall approach of this paper would also work with other methods. The algorithms used by us are described in [22], [34], and [35].

Given an elementary cube Q (not necessarily full) with $Q \subset |\mathcal{X}|$, $\alpha(Q)$ is an interval vector in Ω. Given a time interval $J \subset [0,\infty)$, the aforementioned methods for rigorous numerics compute an interval vector $E \subset \Omega$ such that $\psi(\alpha(Q),J) \subset \operatorname{int}(E)$. Then we represent $q(E) \subset \Sigma$ as a set of cubes in $\mathcal{X}$ as follows:

$$\Phi(Q,J) := \{Q' \in \mathcal{X} \mid p(Q') \cap q(E) \neq \varnothing\} \subset \mathcal{X},$$

which could be infinite because E does not need to be bounded. Then $\phi(p(Q),J) \subset \operatorname{int}[\![\Phi(Q,J)]\!]$. When we denote the restriction of $\Phi(-,J)$ to full cubes in $\mathcal{X}$ by $\mathcal{F}^J$, then $\mathcal{F}^J \colon \mathcal{X} \rightrightarrows \mathcal{X}$ is a combinatorial enclosure of ϕ_t for every $t \in J$. When checking the conditions of Theorem 4.5, we use the intervals $J = [h,h]$, having length zero, and $J = [0,h]$. Our algorithm operates on this finite set of full cubes:

$$\mathcal{K} := \{[i,i+1] \times [j,j+1] \times [k,k+1] \mid i,j,k \in \{0,1,\dots,2^m-1\}\} \subset \mathcal{X}.$$

The dynamics of $\phi(.,J)$ which we are interested in is represented numerically by the function COVER, defined for a cube Q of arbitrary dimension :

$$\text{COVER}(Q,J) := \Phi(Q,J) \cap \mathcal{K}.$$

Definition 6.7. For a set $\mathcal{A} \subset \mathcal{K}$ of full cubes, let

$$\text{WRAP}(\mathcal{A}) := \{Q \in \mathcal{K} \mid Q \cap |\mathcal{A}| \neq \varnothing\},$$

where we consider the right edge at $[2^m]$ glued to the left edge at $[0]$.

Our algorithm does not require more knowledge about the behavior of the dynamical system than the outputs of COVER.

6.3. **Constructing the weak index pair.** We define the following subset of $\mathcal{K}$:

$$\mathcal{M} := \{[i,i+1] \times [j,j+1] \times [k,k+1] \mid 0 \le i < 2^m, j,k \in \{1,\dots,2^m-2\}\}.$$

The geometric realization $[\![\mathcal{M}]\!]$ is our candidate for an isolating neighborhood of ϕ_h. The property $[\![\mathcal{M}]\!] \subset \operatorname{int}[\![\mathcal{K}]\!]$ is crucial for the proof of Proposition 7.5. For a combinatorial enclosure $\mathcal{F} \colon \mathcal{X} \rightrightarrows \mathcal{X}$ of ϕ_h, we let its restriction to $\mathcal{M}$ be defined by

$$\mathcal{F}_{\mathcal{M}} \colon \mathcal{M} \rightrightarrows \mathcal{M}, \quad Q \mapsto \mathcal{F}(Q) \cap \mathcal{M} = \text{COVER}(Q,[h,h]) \cap \mathcal{M}.$$

We use the approach described in [20] to construct a weak index pair for ϕ_h. The algorithm constructs sets of full cubes $\mathcal{N}^3 = \operatorname{Inv}^-(\mathcal{M},\mathcal{F}) = \operatorname{Inv}^-(\mathcal{M},\mathcal{F}_{\mathcal{M}})$ and $\operatorname{Inv}^+(\mathcal{M},\mathcal{F}) = \operatorname{Inv}^+(\mathcal{M},\mathcal{F}_{\mathcal{M}})$ (cf. Definition 6.5). Note that $\operatorname{Inv}([\![\mathcal{M}]\!],\phi_h) \subset [\![\operatorname{Inv}(\mathcal{M},\mathcal{F})]\!]$ in the phase space Σ. Therefore, if

$$\text{WRAP}\left(\operatorname{Inv}^-(\mathcal{M},\mathcal{F}) \cap \operatorname{Inv}^+(\mathcal{M},\mathcal{F})\right) \subset \mathcal{M},$$

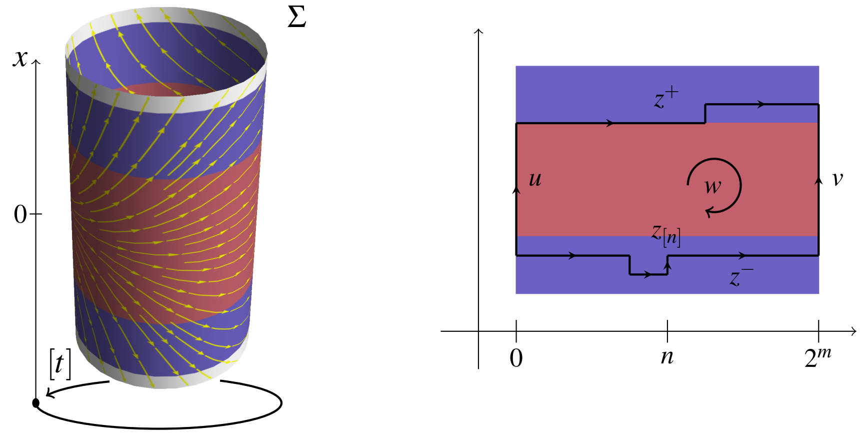


FIGURE 1. A simple example where Σ is just $\mathbb{R}/T\mathbb{Z}\times\mathbb{R}^1$ and assuming the flow ϕ is induced by a vector field $f\colon \mathbb{R}\times\mathbb{R}^1\to\mathbb{R}^1$ with $f(t,0)=0$ and $f(t,x)\cdot x>0$ for $x\neq 0$. The invariant set S of ϕ is the periodic orbit at $x=0$. In the left figure, the blue set is $[\![\mathcal{L}]\!]\subset\Sigma=\mathbb{R}/T\mathbb{Z}\times\mathbb{R}^1$, the blue set in the right figure is the cubical set $|\mathcal{L}|$; similarly for $\mathcal{N}$, the union of the red and blue cubical sets. The right figure shows cubical chains with $\partial w = u - v + z_+ - z_-$ (in the notation of Section 7).

then $[\![\mathcal{M}]\!]$ is an isolating neighborhood for ϕ_h. In this case, we let $\mathcal{L}^3 = \mathrm{Inv}^-(\mathcal{M},\mathcal{F})\setminus \mathrm{Inv}^+(\mathcal{M},\mathcal{F})$. The pair $(\mathcal{N}^3,\mathcal{L}^3)$ of sets of full cubes is called a *combinatorial index pair* and $([\![\mathcal{N}^3]\!],[\![\mathcal{L}^3]\!])$ is a weak index pair for ϕ_h, as shown in [20].

Remark 6.8. After finding the weak index pair, it is also possible to replace $\mathcal{M}$ by a set $\mathcal{M}'\subset\mathcal{M}$ such that $\mathrm{WRAP}(\mathrm{Inv}(\mathcal{M},\mathcal{F}))\subset\mathcal{M}'$. Then $\mathrm{Inv}(\mathcal{M}',\mathcal{F})=\mathrm{Inv}(\mathcal{M},\mathcal{F})$. Using $\mathcal{M}'$, we can reduce the thickness of the exit set $\mathcal{L}^3$.

7. ALGORITHMS

In this section we present the algorithm which constructs a 1-chain v for each given homology generator u such that (u,v) is an h-movable pair of contiguous cycles. The final step to apply Theorem 4.5 then consists of finding a 1-chain homologous to v which is a linear combination of the generators of $H_1(N_0,L_0)$.

Definition 7.1. For an elementary cube $Q=I_1\times I_2\times I_3$, let $\pi_i(Q):=I_i$. Removing the first factor is denoted by $\hat\pi_1$, i.e.,

$$\hat\pi_1(Q):=\pi_2(Q)\times\pi_3(Q)\subset\mathbb{R}^2,$$

We use the definitions of cubical chains and their boundaries from [11], Section 2.2. Here we use the following notations. Let $\mathbb{F}$ be a field and let $\mathcal{A}$ be a finite set of cubes in $\mathbb{R}^3$.

Definition 7.2.

(i) Let $C_k(\mathcal{A}):=\left\{\sum_i \alpha_i Q_i \mid \alpha_i\in\mathbb{F} \text{ and } Q_i\in\mathcal{A}^k\right\}$.
(ii) Let $\mathcal{B}\subset\mathcal{A}$. Define $C_k(\mathcal{A},\mathcal{B}):=\{c\in C_k(\mathcal{A})\mid \partial c\in C_{k-1}(\mathcal{B})\}$.

(iii) For a chain $c = \sum_i \alpha_i Q_i \in C_k(\mathcal{A})$ with $Q_i \neq Q_j$ whenever $i \neq j$, let

$$c(Q) := \begin{cases} \alpha_i \text{ if } Q = Q_i \text{ for some } i, \\ 0 \text{ otherwise.} \end{cases}$$

Definition 7.3.

(i) For $j,k \in \mathbb{Z}$ with $j \leq k$, let

$$\mathcal{A}_{[j,k]} := \{Q \in \mathcal{A} \mid \pi_1(Q) \subset [j,k]\}.$$

Similarly, let $\mathcal{A}_{[j,k)} := \mathcal{A}_{[j,k]} \setminus \mathcal{A}_{[k]}$, $\mathcal{A}_{(j,k]} := \mathcal{A}_{[j,k]} \setminus \mathcal{A}_{[j]}$ and $\mathcal{A}_{(j,k)} := \mathcal{A}_{[j,k)} \setminus \mathcal{A}_{[j]}$.

(ii) For a chain $c = \sum_i \alpha_i Q_i \in C_k(\mathcal{A})$, let $\mathcal{B} = \{Q \in \mathcal{A}^k \mid c(Q) \neq 0\}$. Let I be of the form $[j,k]$, $[j,k)$, $(j,k]$ or (j,k). Then define $c_I := \sum_{Q \in \mathcal{B}_I} c(Q)Q \in C_k(\mathcal{B}_I)$.

(iii) Define $\hat{\pi}_1(c) := \sum_i \alpha_i \hat{\pi}_1(Q)$.

Definition 7.4.

(i) For a chain $c \in C_k(\mathcal{A})$, let its *support* be $|c| := \bigcup\{Q \mid c(Q) \neq 0\}$.

(ii) For $c \in C_k(\mathcal{A})$, let $[\![c]\!] := p(|c|)$.

(iii) We call a 1-chain $c \in C_1(\mathcal{A})$ a *path* if there are $x, y \in \mathcal{A}^0, x \neq y$, such that $\partial c = x - y$.

We assume that the weak index pair $([\![\mathcal{N}^3]\!], [\![\mathcal{L}^3]\!])$ discussed in the previous section was successfully constructed. Now let $\mathcal{N}$ be the smallest set of cubes which contains all boundary cubes of its elements and has the constructed $\mathcal{N}^3$ as a subset. Our notations are then consistent with Definition 6.1. Let $N = [\![\mathcal{N}^3]\!] = [\![\mathcal{N}]\!] \subset \Sigma$. In an analogous way, we define $\mathcal{L}$ and L. Note that in general we only have a subspace relation $[\![\mathcal{N}_{[0]}]\!] \subset [0] \times N_0$, where $[0] \in \mathbb{R}/T\mathbb{Z}$ on the right hand side and $N_0 \subset \mathbb{R}^2$ using notations from Subsection 4.1. In order to compute $H_*(N_0, L_0)$ correctly, we add sets of cubes

$$\{[0] \times I_2 \times I_3 \mid [2^m] \times I_2 \times I_3 \in \mathcal{N}\} \text{ and } \{[2^m] \times I_2 \times I_3 \mid [0] \times I_2 \times I_3 \in \mathcal{N}\}$$

to $\mathcal{N}$. Then $[\![\mathcal{N}_{[0]}]\!] = [\![\mathcal{N}_{[2^m]}]\!] = [0] \times N_0 = [T] \times N_T$ and still $[\![\mathcal{N}]\!] = N$ and $[\![\mathcal{L}]\!] = L$.

Calculating the Conley index in 0-th homology can be done as follows. A generator $[u] \in H_0(N_0, L_0)$ can be represented by a vertex x in a component of $\mathcal{N}_{[0]}$ that has empty intersection with $\mathcal{L}_{[0]}$. The required 1-chain w is a path along edges of $\mathcal{N}$ such that $\phi([\![w]\!], [0,h]) \subset N$ and $\partial w = y - x$ with $y \in \mathcal{N}^0_{[2^m]}$. In our examples, $H_0(N_0, L_0) = 0$.

We are ready to formulate the functions used by our algorithm for computing $\mathrm{CH}(S_0, P)$. Function MOVABLE in Algorithm 1 is the function which executes the integration of ϕ over the time interval $[0,h]$.

function MOVABLE(cube Q, set of full cubes $\mathcal{A}$)
  set of full cubes $\mathcal{B} := \mathrm{COVER}(Q, [0,h])$
  if $\mathcal{B} \subset \mathcal{A}$ **then return** TRUE
  else return FALSE

ALGORITHM 1

Proposition 7.5.

(i) If MOVABLE$(Q, \mathcal{N}^3)$ *is* TRUE *for a cube $Q \in \mathcal{N}$, then $\phi([\![Q]\!], [0,h]) \subset N$.*

(ii) If MOVABLE$(Q, \mathcal{L}^3)$ *is* TRUE *for a cube $Q \in \mathcal{L}$, then $\phi([\![Q]\!], [0,h]) \subset L$.*

Proof.

(i) By assumption, $\textsc{cover}(Q,[0,h]) = \Phi(Q,[0,h]) \cap \mathcal{K} \subset \mathcal{N}^3$. This yields
$$\phi([\![Q]\!],[0,h]) \subset \operatorname{int}[\![\Phi(Q,[0,h]) \cap (\mathcal{K} \cup (\mathcal{X} \setminus \mathcal{K})]\!] \subset [\![\mathcal{N}^3]\!] \cup [\![\mathcal{X} \setminus \mathcal{K}]\!],$$
which is a disjoint union since $\textsc{wrap}(\mathcal{N}^3) \subset \mathcal{K}$. The claim follows from the connectedness of $\phi([\![Q]\!],[0,h])$ and $[\![Q]\!] \subset [\![\mathcal{N}^3]\!]$.

(ii) This is analogous to (i).

□

The functions USABLEN and USABLEL in Algorithm 2 confirm that a cube can be used as a summand of w or z, respectively.

```
function USABLEN(square Q)
    if Q ∉ 𝒩² then return FALSE
    for F ∈ COBOUNDARY(Q) ∩ 𝒩³ do
        for Q' ∈ BOUNDARY(F) do
            if not MOVABLE(Q', 𝒩³) then
                break                                   ▷ try other full cube F
        return TRUE
    return FALSE

function USABLEL(edge e)
    if e ∉ ℒ¹ then return FALSE
    if MOVABLE(e, ℒ³) then
        for Q ∈ COBOUNDARY(e) do
            if USABLEN(Q) then
                return TRUE
    return FALSE

function USABLEN(2-chain c)
    set of squares 𝒜 := {Q | c(Q) ≠ 0}
    if not 𝒜 ⊂ 𝒩² then return FALSE
    for Q ∈ 𝒜 do
        if not USABLEN(Q) then return FALSE
    return TRUE
```

ALGORITHM 2

Lemma 7.6.

(i) If USABLEN(Q) *is* TRUE*, then* $\phi([\![Q]\!],[0,h]) \subset N$.

(ii) If USABLEL(e) *is* TRUE*, then* $\phi([\![e]\!],[0,h]) \subset L$.

(iii) If USABLEN(c) *is* TRUE*, then* $\phi([\![c]\!],[0,h]) \subset N$.

Proof. This follows from Proposition 7.5. □

Remark 7.7. The checks in functions USABLEN and USABLEL are slightly more restrictive than one might expect. Note that USABLEN(Q) is TRUE iff Q is in the boundary of a full cube F for which all boundary cubes are movable. In practice, this helped avoid some dead ends in Algorithm 5.

The symbol VAR in Algorithms 3 and 4 means that the following variable is passed by reference to the function.

For a vertex $y = [i] \times [j] \times [k] \in \mathcal{L}^0$, define the following 5-tuple $S(y)$ of oriented edges

$$\begin{aligned} S(y) \ &= ([i,i+1] \times [j] \times [k], \quad [i] \times [j,j+1] \times [k], \quad [i] \times [j] \times [k,k+1], \\ &\quad -[i] \times [j-1,j] \times [k], \quad -[i] \times [j] \times [k-1,k]), \end{aligned}$$

which is used in Algorithm 3.

1: **function** PATHBACKTRACKING(VAR path c, vertex x)
2:     **if** $c = 0$ **then** $y := x$
3:     **else**
4:         let e be the last edge of c
5:         **if not** USABLEL(e) **then return** FALSE
6:         let y be the endpoint of c
7:     **if** $y \in \mathcal{L}^0_{[2^m]}$ **then return** TRUE
8:     **for** $d \in S(y)$ **do**
9:         1-chain $s := c + d$
10:         **if** PATHBACKTRACKING(s, x) **then**
11:             $c := s$; **return** TRUE   ▷ success
12:     **return** FALSE   ▷ c remains unchanged

13: **function** PATH(vertex x)
14:     1-chain $c_x := 0$
15:     **if** PATHBACKTRACKING(c_x, x) **then return** c_x
16:     **else return** FAILURE

ALGORITHM 3

Proposition 7.8. *Assume that function* PATH *from Algorithm 3 is called with a vertex $x \in \mathcal{L}^0_{[0]}$ as input. If it terminates successfully, then it returns a path c_x satisfying $\partial c_x = y - x$ for some $y \in \mathcal{L}^0_{[2^m]}$ and $\phi([\![c_x]\!], [0,h]) \subset L$.*

Additionally, for every $n \in \{0, \dots, 2^m - 1\}$ there is exactly one edge e such that $\pi_1(e) = [n, n+1]$ and $c_x(e) \neq 0$. This edge has coefficient $c_x(e) = 1$.

Proof. The algorithm performs a depth-first search using backtracking. A new candidate path s is rejected in line 10 if USABLEL$(d) =$ FALSE. Therefore $\phi([\![c_x]\!], [0,h]) \subset L$ follows from Lemma 7.6(ii). The last property follows from the definition of $S(y)$. □

First we use cubical homology software to construct a finite basis $\{[u_j]\}$ of $H_1(\mathcal{N}_{[0]}, \mathcal{L}_{[0]})$, where each $u_j \in C_1(\mathcal{N}_{[0]}, \mathcal{L}_{[0]})$ is a path. From here on we drop the index j for readability and fix some 1-chain $u = u_j$. Then $\partial([0] \times u) = x^+ - x^-$ with $x^+, x^- \in \mathcal{L}^0_{[0]}$. The 2-chain w is constructed by successively adding oriented squares. If necessary, squares within a layer $\mathcal{N}_{[n]}$ are added using the function FLOODFILL in Algorithm 4.

Proposition 7.9. *The function* FLOODFILL *from Algorithm 4 with input $Q \in \mathcal{N}^2_{[n]}$, a chain $c \in C_1(\mathcal{N}_{[n]})$ and $D = 0 \in C_2$ terminates. After execution, D is a 2-chain in $\mathcal{N}_{[n]}$ with $\phi([\![D]\!], [0,h]) \subset N$.*

```
1: function floodFill(square $Q \in \mathcal{N}_{[n]}$, 1-chain $c$, var 2-chain $D$)
2:     if ($D(Q) \neq 0$ or not usableN($Q$)) then return
3:     $D := D + Q$
4:     for edges $e'$ with $(\partial Q)(e') \neq 0$ do        ▷ add neighboring squares of $Q$
5:         if $c(e') \neq 0$ then continue                     ▷ do not cross $c$
6:         for $Q' \in$ coboundary$(e') \cap \mathcal{N}_{[n]}$ do
7:             if not usableN$([n, n+1] \times \partial \hat{\pi}_1(Q'))$ then
8:                 if $Q' \neq Q$ then
9:                     floodFill$(Q', c, D)$
```

ALGORITHM 4

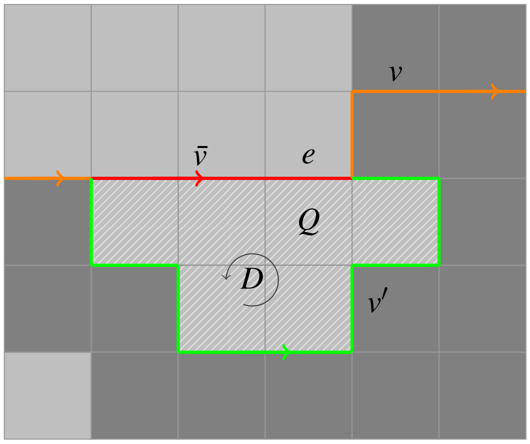
Proof. The recursion terminates because $\mathcal{N}^2_{[n]}$ is finite, hence the search tree is finite. The property $\phi([\![D]\!], [0,h]) \subset N$ is guaranteed by the check in line 2 and Lemma 7.6(i). □

We are ready to formulate Algorithm 5 which constructs v given u. The idea is sketched in Figure 2. Note that the lines containing w could be removed without changing the behavior of the algorithm.

```
function findPartner(1-chain $u$)
    0-chain $x^+ - x^- := \partial u$
    1-chain $z :=$ path$(x^+) -$ path$(x^-)$
    2-chain $w := 0$
    1-chain $v := u + z_{[0]}$
    for $n := 0$ to $2^m - 1$ do
        outerLoopLabel:
        for $e$ with $v(e) \neq 0$ do
            if not usableN$([n, n+1] \times \hat{\pi}_1(e))$ then
                for $Q \in$ coboundary$(e) \cap \mathcal{N}_{[n]}$ do        ▷ try both sides of $e$
                    2-chain $D := 0$
                    floodFill$(Q, v, D)$
                    if $(\partial Q)(e) \cdot v(e) = 1$ then $D := -D$     ▷ switch orientation
                    1-chain $\bar{v} := \sum_{v(e')(\partial D)(e') \neq 0} v(e')e'$
                    1-chain $v' := \bar{v} + \partial D$
                    if usableN$([n, n+1] \times \hat{\pi}_1(v'))$ then
                        $w := w - D$
                        $v := v - \bar{v} + v'$
                        goto outerLoopLabel
                return failure        ▷ give up if adding squares in $\mathcal{N}_{[n]}$ did not help
        $w := w - [n, n+1] \times \hat{\pi}_1(v)$
        $v := [n+1] \times \hat{\pi}_1(v) + z_{[n+1]}$
    return $v$
```

ALGORITHM 5

Definition 7.10. Let $i, j \in \mathbb{N}$ and $i \leq j$. A pair (u, v) of cubical cycles $u \in C_1(\mathcal{N}_{[i]}, \mathcal{L}_{[i]})$, $v \in C_1(\mathcal{N}_{[j]}, \mathcal{L}_{[j]})$ is called *contiguous and movable* if the corresponding pair of singular



The squares drawn are in $\{Q' \in \mathcal{N}_{[n]} \mid \text{USABLEN}(Q') = \text{TRUE}\}$. Imagine the time axis (n axis) perpendicular to the figure. Each square Q' is dark gray if $\text{USABLEN}([n, n+1] \times \partial \hat{\pi}_1(Q'))$ is TRUE and light gray otherwise. Line 7 of Alg. 4 shows that FLOODFILL adds light gray squares to D (hatched area). The red 1-chain $\bar{v}$ is a part of v (red and orange) that is replaced by the homologous 1-chain v' (green).

FIGURE 2. A typical state of the variables of Algorithm 5 in line 15

cycles is a contiguous pair of h-movable cycles over $[iT/2^m, jT/2^m]$ as defined in Subsection 4.2.

Proposition 7.11. *When function* FINDPARTNER *from Algorithm 5 is run with input a path $u \in C_1(\mathcal{N}_{[0]}, \mathcal{L}_{[0]})$ and it returns v, then (u, v) is a contiguous and movable pair of cubical cycles. Additionally, $\partial w = u - v + z$.*

Proof. Proposition 7.8 and the definition of z in line 3 show that $\phi([\![z]\!], [0, h]) \subset L$. Note that right after the initialization of v in line 5, the pair (u, v) is contiguous and movable because $u - v + z_{[0]} = 0$. Then for every change of the variable v, let v_{old} be its old value and v_{new} its new value. The proposition is proven by showing that the pair $(v_{\text{old}}, v_{\text{new}})$ is contiguous and movable at every change (cf. Lemma 4.3). There are two kinds:

(i) The change in line 18: Observe that $\partial D = v_{\text{new}} - v_{\text{old}}$ and $\phi([\![D]\!], [0, h]) \subset N$ because it was constructed using FLOODFILL (cf. Proposition 7.9)

(ii) The change in line 22: The successful termination of the for-loop in line 8 together with the check in line 9 ensures that $\phi([\![[n, n+1] \times \hat{\pi}_1(v_{\text{old}})]\!], [0, h]) \subset N$ by Lemma 7.6(i). Additionally,

$$\begin{aligned} \partial\left([n, n+1] \times \hat{\pi}_1(v_{\text{old}})\right) &= \partial[n, n+1] \times \hat{\pi}_1(v_{\text{old}}) - [n, n+1] \times \partial \hat{\pi}_1(v_{\text{old}}) \\ &= [n+1] \times \hat{\pi}_1(v_{\text{old}}) - [n] \times \hat{\pi}_1(v_{\text{old}}) - z_{(n,n+1)} = v_{\text{new}} - v_{\text{old}} - z_{(n,n+1]}. \end{aligned}$$

The property $\partial w = u - v + z$ follows from adding these equations over all changes of w. □

Proposition 7.11 and Theorem 4.5 now yield:

Theorem 7.12. *Let $T/h \in \mathbb{Q}$ and let $(\mathcal{N}, \mathcal{L})$ be sets of cubes constructed as above, in particular $([\![N]\!], [\![L]\!])$ is a weak index pair for (S, ϕ_h).*

When the function FINDMATRIX *from Algorithm 6 does not fail and returns A, then* $\text{CH}(S_0, P)$ *is equal to the conjugacy class of RA.*

Proof. Observe that each pair $(v_j, \sum_i a_{ij} \bar{u}_i)$ is contiguous and movable by construction in Algorithm 6. Since all pairs (u_j, v_j) are contiguous and movable by Proposition 7.11, all pairs $(u_j, \sum_i a_{ij} \bar{u}_i)$ are contiguous and movable by Lemma 4.3. □

function FINDMATRIX(sets of full cubes $\mathcal{N}, \mathcal{L}$)
  construct $\{u_1, \ldots, u_k\} \subset C_1(\mathcal{N}_{[0]}, \mathcal{L}_{[0]})$ representing a basis of $H_1(\mathcal{N}_{[0]}, \mathcal{L}_{[0]})$
  $\bar{u}_i := [2^m] \times \hat{\pi}_1(u_i)$ for each i
  $(k \times k)$-matrix $A = [a_{ij}]$, every $a_{ij} := 0 \in \mathbb{F}$.
  for $j := 1$ **to** k **do**
    $v :=$ FINDPARTNER(u_j)
    construct $w' \in C_2(\mathcal{N}_{[2^m]})$ such that: ▷ analogous to Alg. 5, lines 7 to 20
      (i) MOVABLE$(Q, \mathcal{N}^3)$ whenever $w'(Q) \neq 0$;
      (ii) if $(v - \partial w')(e) \neq 0$, then MOVABLE$(e, \mathcal{L}^3)$ or $u_i(e) \neq 0$ for some i; and
      (iii) there is an edge e such that $v(e)w'(e) = 1$.
    $v' := v - \partial w'$
    for $i := 1$ **to** k **do** ▷ fill column j of matrix A
      find $e \in \mathcal{N}^1_{[2^m]}$ such that $\bar{u}_i(e) \cdot v'(e) \neq 0$
      if such an e was found **then** $a_{ij} := v'(e)/\bar{u}_i(e)$
    while $c := v' - \sum_i a_{ij}\bar{u}_i \neq 0$ **do** ▷ check if column j of A is correct
      let e be an edge with $c(e) \neq 0$
      if MOVABLE$(e, \mathcal{L})$ **then** $v' := v' - c(e)$
      else return FAILURE
  return A

ALGORITHM 6

8. EXAMPLES

8.1. **One-dimensional first relative homology.** We applied Algorithm 6 using the weak index pair constructed as in Subsection 6.3 to the differential equation

$$\dot{z} = (1 + e^{i\eta t}|z|^2)\bar{z},$$

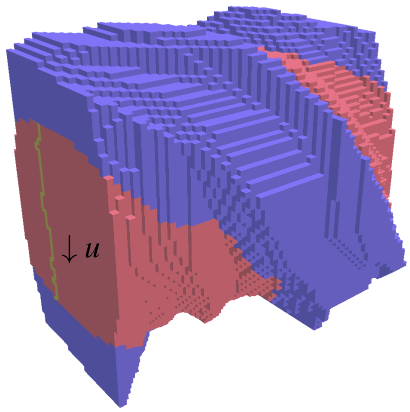
which shows chaotic behavior for $\eta \in (0, 1]$ (cf. [21] and references therein). This equation has period $T = 2\pi/\eta$ in t. We analyzed the equation for $\eta = 2.0$ using the parameter $h = 1/64 = 0.015625$. More precisely, since π is irrational, the algorithm used $\pi' \in [\pi - \varepsilon, \pi + \varepsilon]$, where ε is the machine precision. Therefore $T' = 2\pi'/\eta \in \mathbb{Q}$, i.e., the numerical proof is found for $\eta' = 2\pi'/T'$ instead of η. We covered the candidate $M = S^1 \times [-3, 3] \times [-3, 3] = [\![\mathcal{M}]\!]$ for an isolating neighborhood using cubes of equal size as described above. Our algorithm found a combinatorial index pair $(\mathcal{N}, \mathcal{L})$ inside $\mathcal{M}$ at a subdivision depth of $m = 6$. The chains constructed by our algorithm are shown in Figure 3. The output was $A = [-1]$.

Theorem 7.12 applies. We conclude that the generator $[u] \in H_1(N_0, L_0)$ is sent to $-[u] \in H_1(N_0, L_0)$ under the relative homology endomorphism induced by the Poincaré map P.

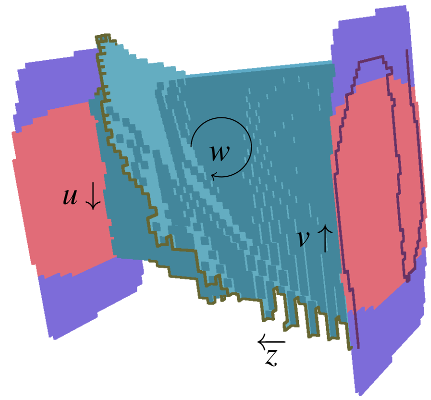
Finding the combinatorial index pair $(\mathcal{N}, \mathcal{L})$ took 330 seconds seconds using a 2.4GHz CPU with 6 GB of RAM. The construction of all the chains in Algorithm 6 took 149 seconds. Most of the time was used for the rigorous integrations. The set $\mathcal{K}$ consists of $(2^6)^3 = 262\,144$ cubes, the set $\mathcal{N}$ of $132\,728$ cubes.

8.2. **Two-dimensional first relative homology.** We applied the same algorithms as above to the equation

$$\dot{z} = e^{i\eta t}\bar{z}^2 + \bar{z},$$

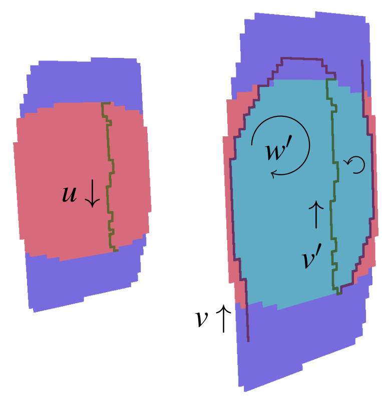


(A) The full cubes in $(\mathcal{N},\mathcal{L})$ (blue: $\mathcal{L}^3$, red: $\mathcal{N}^3 \setminus \mathcal{L}^3$) and u on the left-hand side.



(B) The blue squares are $\mathcal{L}^2_{[0]} \cup \mathcal{L}^2_{[2^m]}$. Similarly for $\mathcal{N}$. The chains after executing FINDPARTNER(u) are shown.

FIGURE 3. The intermediate results of the example in Section 8.1. The algorithm makes sure that $\partial w = u - v + z$ as shown in Figure 3B. Figure 3C shows the construction of v' from v. Note that $\partial w' - v + v'$ lies in $\mathcal{L}$. The movability properties of the chains cannot be seen in the figure, but are checked numerically. The output is $A = [-1]$, which can be seen from Figure 3C because u points down and v' points up.



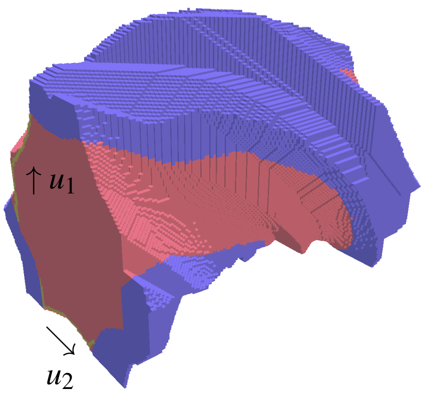
(C) Overall results. In this special example, the support of v' is contained in the support of v.

This equation has period $T = 2\pi/\eta$ in t. We analyzed the equation for $\eta = 2.0$. Again, we used the candidate $M = S^1 \times [-3,3] \times [-3,3] = [\![\mathcal{M}]\!]$ for an isolating neighborhood. Our software found a combinatorial index pair $(\mathcal{N},\mathcal{L})$ inside $\mathcal{M}$ at a refinement depth of $m = 7$ using the parameter $h = 1/64$. The index pair with the resulting chains is shown in Figure 4. Using Theorem 7.12, the Conley index of the Poincaré map is given up to conjugacy by

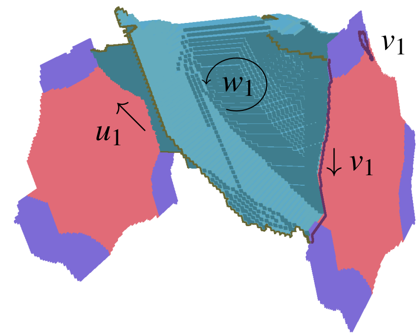
$$\mathrm{CH}(S_0,P) = RA = R\left(\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Finding $(\mathcal{N},\mathcal{L})$ took 2256 seconds on the same hardware as before. Then Algorithm 6 required further 545 seconds.

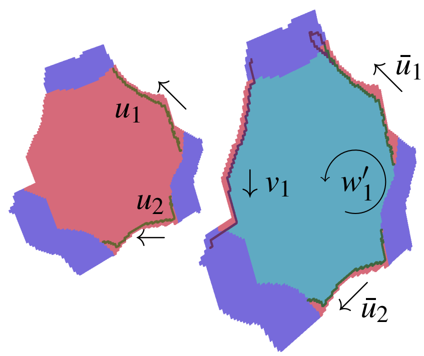
In contrast to this, when starting with $M = S^1 \times [-0.1,0.1] \times [-0.1,0.1]$, the algorithm yields a different Conley index because the output in this case is $A = [1]$. Since the Conley index is a function of the invariant sets, $\mathrm{Inv}([-0.1,0.1]^2,P) \neq \mathrm{Inv}([-3,3]^2,P)$.



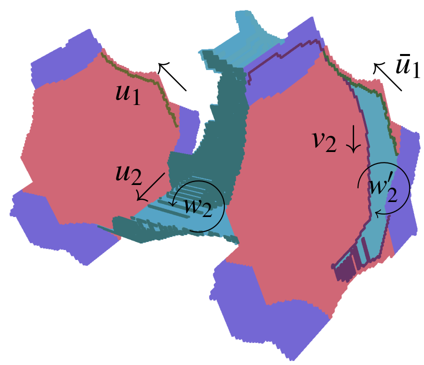
(A) Index pair $(\mathcal{N},\mathcal{L})$ and 1-chains u_1, u_2



(B) The construction of v_1. Note that v_1 consists of two connected 1-chains, one of which has boundary 0.



(C) The 2-chain w'_1 fills most of $\mathcal{N}_{[2^m]} \setminus \mathcal{L}_{[2^m]}$. Here $v'_1 = -\bar{u}_1 + \bar{u}_2$.



(D) Computing the second column of A. Here $v'_2 = -\bar{u}_1$.

FIGURE 4. Outputs for the example in Subsection 8.2, using the same colors as in Figure 3

9. CONCLUDING REMARKS

The theory and algorithms presented in this paper show that the Conley index of a Poincaré map may be computed algorithmically without the need of the, often prohibitive and always computationally very expensive, long time rigorous integration. Moreover, the proposed method uses general algorithms for the construction of index pairs applied to the time h map of the flow and, in consequence, is fully automatic. In particular, it does not require manual construction of isolating blocks as in the case of the method presented in [21].

In this paper the method, particularly Theorem 4.5, is only developed for periodic non-autonomous equations. By their very nature, these systems yield easily describable Poincaré sections – the hyperplanes for fixed $[t] \in S^1$. An analogous method for Poincaré maps of periodic orbits of arbitrary autonomous ODEs is more challenging but we are convinced it is possible and definitely very desirable. We leave it as the subject of subsequent research.

The idea of the algorithms in Section 7 also applies to higher-dimensional spaces. If we work on the cylinder $\Sigma = \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^d$ and we are interested in the Conley index $\mathrm{CH}(S_0, P)$ for $d \geq 3$, then the generalization is straightforward: The set $\mathcal{N}_{[n]}$ then represents a d-dimensional hyperplane, and the function FLOODFILL should be adapted to recursively add squares in $\mathcal{N}^2_{[n]}$. For $d = 3$, the intermediate steps for each n could easily be visualized. Larger d are harder to handle because the runtime and the memory of our integration algorithms grow exponentially with dimension.

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