ESTIMATES OF THE BERGMAN DISTANCE ON DINI-SMOOTH BOUNDED PLANAR DOMAINS

NIKOLAI NIKOLOV AND MARIA TRYBUŁA

Abstract. Precise estimates for the Bergman distances of Dini-smooth bounded planar domains are given. These estimates imply that on such domains the Bergman distance almost coincides with the Carathéodory and Kobayashi distances.

1. Results

In [6, Proposition 8], the first named author found optimal estimates for Carathéodory and Kobayashi distances, c_D and k_D, on Dini-smooth bounded planar domains D in terms of $d_D = \operatorname{dist}(\cdot, \partial D)$. In this paper we shall prove similar estimates for the Bergman distance b_D. For convenience of the reader, the definitions of these three distances, as well as of Dini-smoothness, are given in the next section.

Proposition 1. *Let D be a Dini-smooth bounded planar domain. Then there exists a constant $c > 1$ such that*

$$\sqrt{2}\log\left(1+\frac{|z-w|}{c\sqrt{d_D(z)d_D(w)}}\right) \le b_D(z,w)$$

$$\le \sqrt{2}\log\left(1+\frac{c|z-w|}{\sqrt{d_D(z)d_D(w)}}\right), \quad z,\, w \in D.$$

By [6, Proposition 8], the same result holds for $\sqrt{2}c_D$ and $\sqrt{2}k_D$ instead of b_D. So, we have the following

2010 *Mathematics Subject Classification.* 32A25, 32F45.

Key words and phrases. Bergman, Carathéodory and Kobayashi distances; Dini-smooth planar domain.

M. Trybuła is partially supported by the International PhD programme "Geometry and Topology in Physical Models" of the Foundation for Polish Science, by the Polish National Science Center – grant PRO-2013/11/N/ST1/03609, and by the Bulgarian National Science Found under contract DFNI-I 02/14. The initial version of this paper was prepared during her visit to the Institute of Mathematics and Informatics, Bulgarian Academy of Science, October 2013 – April 2014 and July 2014.

Corollary 2. *If D is a Dini-smooth bounded planar domain, then the differences $b_D - \sqrt{2}c_D$ and $b_D - \sqrt{2}k_D$ are bounded.*

Note that Proposition 1 is equivalent to

Proposition 1′. *Let D be a Dini-smooth bounded planar domain. There exists a constant $c > 1$ such that:*

- *if $|z-w|^2 > d_D(z)d_D(w)$, then*

$$\log\frac{|z-w|^2}{d_D(z)d_D(w)} - c < \sqrt{2}b_D(z,w) < \log\frac{|z-w|^2}{d_D(z)d_D(w)} + c;$$

- *if $|z-w|^2 \le d_D(z)d_D(w)$, then*

$$\frac{|z-w|}{c\sqrt{d_D(z)d_D(w)}} \le b_D(z,w) \le \frac{c|z-w|}{\sqrt{d_D(z)d_D(w)}}.$$

Remark. (a) The Dini-smoothness is essential as an example of a $\mathcal{C}^1$-smooth bounded simply connected planar domain shows (see [8, Example 2]).

(b) One of the missing properties of b_D in comparison with c_D and l_D is monotonicity under inclusion of (planar) domains. However, the invariants M_D and K_D share this property which allows us to modify the approach from [6].

(c) Results in $\mathbb{C}^n$ in the spirit of Proposition 1 and Corollary 3 can be found in [1] and [7], respectively, where the strictly pseudoconvex domains are treated. Note also that the Levi pseudoconvex corank one domains are considered in [3]. As can be expected, our estimates are more precise than those in [1] and [3], when the two points, z and w, are close to each other.

(d) It follows by the second statement of Proposition 1′ that

$$\frac{1}{cd_D(u)} \le \liminf_{\substack{z,w\to u\\ z\neq w}} \frac{b_D(z,w)}{|z-w|}, \quad u \in D.$$

This inequality agrees with the fact that (cf. [5, Lemma 4.3.3 (e)])

$$\limsup_{\substack{z,w\to u\\ z\neq w}} \frac{b_D(z,w)}{|z-w|} \le \beta_D(u;1)$$

(cf. [5, Lemma 4.3.3 (e)]) and the equality (see [4, Remark, p. 11])

$$\lim_{u\to\partial D} \beta_D(u;1)d_D(u) = \frac{\sqrt{2}}{2}.$$

Recall now another comparison result between c_D and k_D (see [6, Proposition 9]): if D is a finitely connected bounded planar domain without isolated boundary points,[1] then

$$\lim_{\substack{w\to\partial D\\ z\neq w}} \frac{c_D(z,w)}{k_D(z,w)} = 1 \quad \text{uniformly in } z \in D. \tag{1}$$

Similar results for c_D, k_D, l_D and b_D in the strictly pseudoconvex case can be found in [9, Theorem 1] and [7, Proposition 4].

The next proposition shows that (1) remains true if we replace c_D or k_D by $b_D/\sqrt{2}$.

Proposition 3. *If D is a finitely connected bounded planar domain without isolated boundary points, then*

$$\lim_{\substack{w\to\partial D\\ z\neq w}} \frac{b_D(z,w)}{c_D(z,w)} = \lim_{\substack{w\to\partial D\\ z\neq w}} \frac{b_D(z,w)}{k_D(z,w)} = \sqrt{2} \quad \text{uniformly in } z \in D.$$

Remark. The isolated boundary points condition is essential. Indeed, if p is an isolated boundary point of a planar domain $D \neq \mathbb{C}\setminus\{p\}$, then $c_D = c_{D\cup\{p\}}$ and $b_D = b_{D\cup\{p\}}$, but $k_D(z,w) \to \infty$ as $w \to p$ and $z \in D$ is fixed.

2. Definitions

1. A boundary point p of a planar domain D is said to be Dini-smooth if ∂D near p is given by a Dini-smooth curve $\gamma : [0,1] \to \mathbb{C}$ with $\gamma' \neq 0$ (i.e., $\int_0^1 \frac{\omega(t)}{t} dt < \infty$, where ω is the modulus of continuity of γ'). A planar domain is called Dini-smooth if all its boundary points are Dini-smooth.

2. Let D be a domain in $\mathbb{C}^n$.

The Bergman distance b_D of D is the integrated form of the Bergman metric β_D, i.e.,

$$b_D(z,w) = \inf_\gamma \int_0^1 \beta_D(\gamma(t);\gamma'(t))dt, \quad z,w \in D,$$

where the infimum is taken over all smooth curves $\gamma : [0,1] \to D$ with $\gamma(0) = z$ and $\gamma(1) = w$.

Recall that

$$\beta_D(z;X) = \frac{M_D(z;X)}{K_D(z)}, \quad z \in D,\ X \in \mathbb{C}^n,$$

[1] Any C^1-smooth bounded planar domain is such a domain.

where

$$M_D(z;X) = \sup\{|f'(z)X| : f \in L_h^2(D),\ \|f\|_{L^2(D)} \le 1,\ f(z)=0\}$$

and

$$K_D(z) = \sup\{|f(z)| : f \in L_h^2(D),\ \|f\|_{L^2(D)} \le 1\}$$

is the square root of the Bergman kernel on the diagonal (we assume that $K_D > 0$; for example, this holds if D is bounded).

The Carathéodory distance c_D and the Lempert function l_D of D are defined as follows:

$$c_D(z,w) = \sup\{\tanh^{-1}|f(w)| : f \in \mathcal{O}(D,\mathbb{D}),\ \text{ with } f(z)=0\},$$

$$l_D(z,w) = \inf\{\tanh^{-1}|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D},D) \text{ with } \varphi(0)=z, \varphi(\alpha)=w\},$$

where $\mathbb{D}$ is the unit disc.

The Kobayashi distance k_D is the largest pseudodistance not exceeding l_D. It is well-known that k_D is the integrated form of Kobayashi metric κ_D defined by

$$\kappa_D(z;X) = \inf\{|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D},D) \text{ with } \varphi(0)=z,\ \alpha\varphi'(0)=X\}.$$

Note that $c_D \le k_D \le l_D$ and $c_D \le b_D$. On the other hand, $k_D = l_D$ for any planar domain D (cf. [5, Remark 3.3.8(e)]).

We refer to [5] for other basic properties of the above invariants.

3. Proofs

To prove Proposition 1, we shall need the following

Lemma 4. *(a)*

$$\log\left(1+\frac{|z-w|}{\sqrt{(1-|z|^2)(1-|w|^2)}}\right) \le \frac{b_{\mathbb{D}}(z,w)}{\sqrt{2}}$$
$$\le \log\left(1+\frac{2|z-w|}{\sqrt{(1-|z|^2)(1-|w|^2)}}\right);$$

(b)

$$\log\left(1+\frac{|z-w|}{2\sqrt{d_{\mathbb{D}}(z)d_{\mathbb{D}}(w)}}\right) \le \frac{b_{\mathbb{D}}(z,w)}{\sqrt{2}} \le \log\left(1+\frac{\sqrt{2}|z-w|}{\sqrt{d_{\mathbb{D}}(z)d_{\mathbb{D}}(w).}}\right).$$

Proof. (a) We have that

$$\sqrt{2}b_{\mathbb{D}}(z,w) = 2k_{\mathbb{D}}(z,w) = \log\frac{1+\left|\dfrac{z-w}{1-\bar{z}w}\right|}{1-\left|\dfrac{z-w}{1-\bar{z}w}\right|} =$$

$$\log\left(1+\frac{2|z-w|}{|1-\bar{z}w|-|z-w|}\right)=\log\left(1+2|z-w|\frac{|1-\bar{z}w|+|z-w|}{(1-|z|^2)(1-|w|^2)}\right).$$

It remains to use that

$$|1-\bar{z}w|^2=(1-|z|^2)(1-|w|^2)+|z-w|^2 \tag{2}$$

and hence

$$\sqrt{(1-|z|^2)(1-|w|^2)}\le|1-\bar{z}w|\le\sqrt{(1-|z|^2)(1-|w|^2)}+|z-w|.$$

(b) The lower estimate follows from (a) and $d_{\mathbb{D}}(z)=1-|z|\ge\dfrac{1-|z|^2}{2}$.

To get the upper estimate, we have to show that

$$1+2|z-w|\frac{|1-\bar{z}w|+|z-w|}{(1-|z|^2)(1-|w|^2)}\le\left(1+\frac{\sqrt{2}|z-w|}{\sqrt{(1-|z|)(1-|w|)}}\right)^2$$

which is equivalent to

$$\frac{|1-\bar{z}w|+|z-w|}{(1-|z|^2)(1-|w|^2)}\le\frac{\sqrt{2}}{\sqrt{(1-|z|)(1-|w|)}}+\frac{|z-w|}{(1-|z|)(1-|w|)},$$

i.e.,

$$|1-\bar{z}w|\le\sqrt{2(1-|z|)(1-|w|)}(1+|z|)(1+|w|)+|z-w|(|z|+|w|+|zw|).$$

So, it is enough to prove that

$$|1-\bar{z}w|^2\le 2(1-|z|)(1-|w|)(1+|z|)^2(1+|w|)^2+|z-w|^2(|z|+|w|+|zw|)^2.$$

Using (2) and dividing by $(1+|z|)(1+|w|)$, the last inequality becomes

$$|z-w|^2(1-|z|-|w|-|zw|)\le(1-|z|-|w|+|zw|)(1+2|z|+2|w|+2|zw|)$$

which follows from $|z-w|^2\le|z|^2+|w|^2+2|zw|\le|z|+|w|+2|zw|$.

Remark. (a) The constants 1 and 2 in front of $|z-w|$ in the lower and upper estimates in Lemma 4 (a) are sharp. To see this, let

$$\frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}\to 0 \text{ and } \infty, \text{ respectively.}$$

(b) The constants $\dfrac{1}{2}$ and $\sqrt{2}$ in front of $|z-w|$ in the lower and upper estimates in Lemma 4 (b) are sharp, too. To see this, let $|z|\to 1$ and then

$$\frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}\to 0 \text{ and } w\to 0, \text{ respectively.}$$

Proof of Proposition 1. Let $D\supset(z_n)\to p$ and $D\supset(w_n)\to q$ ($z_n\ne w_n$). It is enough to find a constant $c>1$ such that the respective estimates for $b_D(z_n,w_n)$ hold for any n.

Note that, by [6, Proposition 5 and Corollary 6], for any neighborhood U of p there exist a neighborhood $V \subset U$ and a constant $c_1 > 0$ such that

$$|\sqrt{2}b_D(z,w) + \log d_D(z) + \log d_D(w)| < c_1, \quad z \in D \cap V, w \in D \setminus U. \tag{3}$$

This inequality provides the desired constant if $D \ni p \neq q \in D$, or $p \in \partial D$, $q \in D$, or $p \in D$, $q \in \partial D$, or $\partial D \ni p \neq q \in \partial D$.

For a planar domain Ω, set $\beta_\Omega(z) = \beta_\Omega(z;1)$, $M_\Omega(z) = M_\Omega(z;1)$ and $\kappa_\Omega(z) = \kappa_\Omega(z;1)$.

If $p = q \in D$, then the continuity of β_D implies that

$$\frac{b_D(z_n,w_n)}{|z_n - w_n|} \to \beta_D(p) > 0$$

and we may easily find the desired constant.

It remains to consider the most difficult case $p = q \in \partial D$. Some of our arguments will be close to that in the proof of [6, Proposition 5].

This proof allows us to assume that $p = 1$ and

$$\{z \in \mathbb{D} : |z-1| < r\} =: E_r \subset D \subset \mathbb{D}$$

for some $r > 0$ (after an appropriate conformal map). Then

$$\sqrt{2}\frac{\kappa^2_{\mathbb{D}}(z)}{\kappa_{E_r}(z)} = \frac{M_{\mathbb{D}}(z)}{K_{E_r}(z)} \le \beta_D(z) \le \frac{M_{E_r}(z)}{K_{\mathbb{D}}(z)} = \sqrt{2}\frac{\kappa^2_{E_r}(z)}{\kappa_{\mathbb{D}}(z)}, \quad z \in E_r \tag{4}$$

(the both equalities hold because E_r is a simply connected domain).

Fix an $r_1 \in (0,r)$. The localization of the Kobayashi metrics from [2, Theorem 2.1 and Lemma 2.2] implies that

$$\kappa_{\mathbb{D}}(z) > (1 - c_2 d_{\mathbb{D}}(z))\kappa_{E_r}(z), \quad z \in E_{r_1}, \tag{5}$$

for some constant $c_2 > 0$. Choose an $r_2 \in (0, r_1]$ with $2c_2 r_2 \le 1$. Then

$$(1 - c_2 d_{\mathbb{D}}(z))\kappa_{\mathbb{D}}(z) < \frac{\beta_D(z)}{\sqrt{2}} < (1 + 2c_2 d_{\mathbb{D}}(z))\kappa_{\mathbb{D}}(z), \quad z \in E_{r_2}.$$

Since $\kappa_{\mathbb{D}}(z) = \dfrac{\beta_{\mathbb{D}}(z)}{\sqrt{2}} = \dfrac{1}{1-|z|^2}$, it follows for $c_3 = 2\sqrt{2}c_2$ that

$$\frac{1}{\sqrt{2}}\left(\frac{1}{d_{\mathbb{D}}(z)} - c_2\right) < \beta_D(z) < \beta_{\mathbb{D}}(z) + c_3, \quad z \in E_{r_2}. \tag{6}$$

We may assume that $z_n, w_n \in E_{r_3}$, where $r_3 \in (0, r_2/2]$ is such that if α_n is the shorter arc with endpoints z_n and w_n of the circle through z_n and w_n which is orthogonal to the unit circle, then $\alpha_n \subset E_{r_2}$. Hence

$$b_D(z_n, w_n) < \int_{\alpha_n} \left(\frac{\sqrt{2}}{1-|z|^2} + c_3\right) dl$$

$$= b_{\mathbb{D}}(z_n, w_n) + c_3 l(\alpha_n) < b_{\mathbb{D}}(z_n, w_n) + 2c_3|z_n - w_n|$$

for any n.

Now, using Lemma 4 (b) and the equality

$$d_{\mathbb{D}}(z) = d_D(z), \quad z \in E_{r_3}, \tag{7}$$

it is easy to find a constant $c > 1$ such that the upper estimate for $b_D(z_n, w_n)$ in Proposition 1 holds for any n.

It is left to manage the lower estimate. Let $\gamma_n : [0,1] \to D$ be a smooth curve such that $\gamma_n(0) = z_n$, $\gamma_n(1) = w_n$ and

$$b_D(z_n, w_n) + |z_n - w_n| > \int_0^1 \beta_D(\gamma_n(t); \gamma_n'(t))dt.$$

Consider the set A of all n for which $\gamma_n(0,1) \not\subset E_{r_2}$. For any $n \in A$ we may find a number $t_n \in (0,1)$ such that $|u_n - 1| = r_2$, where $u_n = \gamma(t_n)$. By (3), there exists a constant $c_4 > 0$, which does not depend on $n \in A$, such that

$$b_D(z_n, w_n) + |z_n - w_n| > b_D(z_n, u_n) + b_D(u_n, w_n)$$
$$> -\frac{\log d_D(z_n)}{\sqrt{2}} - \frac{\log d_D(w_n)}{\sqrt{2}} - c_4.$$

This inequality easily provides a constant $c > 1$ for which the lower estimate for $b_D(z_n, w_n)$ in Proposition 1 holds for any $n \in A$.

Let now $n \notin A$. Since

$$d_{\mathbb{D}}(\gamma_n(t)) \le f_n(t) := d_{\mathbb{D}}(z_n) + |z_n - \gamma_n(t)| < 2r_3 + r_2 \le 2r_2 \le 1/c_2,$$
$$d_{\mathbb{D}}(\gamma_n(t)) \le g_n(t) := d_{\mathbb{D}}(w_n) + |w_n - \gamma_n(t)| < 1/c_2$$

and $|s|' \ge |s'|$, it follows by (6) that, for any $t_n \in (0,1)$,

$$\sqrt{2}(b_D(z_n, w_n) + |z_n - w_n|) > \int_0^1 \left(\frac{1}{d_{\mathbb{D}}(\gamma_n(t))} - c_2 \right) |\gamma_n'(t)| dt$$
$$\ge \int_0^{t_n} \left(\frac{1}{f_n(t)} - c_2 \right) df_n(t) - \int_{t_n}^1 \left(\frac{1}{g_n(t)} - c_2 \right) dg_n(t)$$
$$= \log \left(1 + \frac{|z_n - \gamma_n(t_n)|}{d_{\mathbb{D}}(z_n)} \right) - c_2|z_n - \gamma_n(t_n)|$$
$$+ \log \left(1 + \frac{|w_n - \gamma_n(t_n)|}{d_{\mathbb{D}}(w_n)} \right) - c_2|w_n - \gamma_n(t_n)|$$
$$> \log \left(1 + \frac{|z_n - \gamma_n(t_n)|.|w_n - \gamma_n(t_n)|}{c_5 d_{\mathbb{D}}(z_n) d_{\mathbb{D}}(w_n)} \right)$$

for some constant $c_5 > 1$. Choosing now t_n such that $|z_n - \gamma_n(t_n)| = |w_n - \gamma_n(t_n)|$ and using (7), we obtain the lower estimate in Proposition 1.

So, Proposition 1 is completely proved.

Proof of Proposition 3. By the Köbe uniformization theorem, we may assume that ∂D consists of disjoint circles. Using Proposition 1, Corollary 2, (1) and compactness, it is enough to prove that

$$\lim_{\substack{z,w\to p\\ z\neq w}}\frac{b_D(z,w)}{k_D(z,w)}=\sqrt{2}$$

for any point $p\in\partial D$.

Applying an inversion, we may assume that the outer boundary of D is the unit circle and $p=1$. Then (4) and (5) imply

$$\lim_{z\to 1}\frac{\beta_{E_r}(z)}{\beta_D(z)}=1=\lim_{z\to 1}\frac{\kappa_{E_r}(z)}{\kappa_D(z)}.$$

The first equality shows that $\liminf\limits_{\substack{z,w\to 1\\ z\neq w}}\dfrac{b_{E_r}(z,w)}{b_D(z,w)}\geq 1.$

To get that

$$\limsup_{\substack{z,w\to 1\\ z\neq w}}\frac{b_{E_r}(z,w)}{b_D(z,w)}\leq 1, \tag{8}$$

we shall follow the proof of [9, Proposition 3]. Fix an $\varepsilon>0$ and choose an $r_1\in(0,r)$ such that

$$\beta_{E_r}(z)<(1+\varepsilon)\beta_D(z),\quad z\in E_{r_1}.$$

Combining the argument in the case $n\notin A$ from the previous proof and the estimates from Proposition 1, we may find an $r_2\in(0,r_1)$ such that if $z,w\in E_{r_2}$ and $\gamma:[0,1]\to D$ is a smooth curve for which $\gamma(0)=1$, $\gamma(1)=w$ and

$$\int_0^1\beta_D(\gamma(t);\gamma'(t))dt\leq(1+\varepsilon)b_D(z,w),$$

then $\gamma([0,1])\subset E_{r_1}$. It follows that

$$b_{E_r}(z,w)\leq\int_0^1\beta_{E_r}(\gamma(t);\gamma'(t))dt$$

$$\leq(1+\varepsilon)\int_0^1\beta_D(\gamma(t);\gamma'(t))dt\leq(1+\varepsilon)^2b_D(z,w),\quad z,w\in E_{r_2}.$$

To obtain (8), it remains to let $\varepsilon\to 0$.

So, $\lim\limits_{\substack{z,w\to 1\\ z\neq w}}\dfrac{b_{E_r}(z,w)}{b_D(z,w)}=1.$

On the other hand, [6, Proposition 8] gives the estimates from Proposition 1 for $2k_D$ instead of $\sqrt{2}b_D$. Then we obtain as above

$$\lim_{\substack{z,w\to 1\\ z\neq w}} \frac{\kappa_{E_r}(z,w)}{\kappa_D(z,w)} = 1.$$

Now, the equality $b_{E_r} = \sqrt{2}k_{E_r}$ completes the proof.

References

[1] Z. M. Balogh, M. Bonk, *Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains*, Comment. Math. Helv. 75 (2000), 504–533.

[2] F. Forstneric, J.-P. Rosay *Localization ot the Kobayashi metric and the boundary continuity of proper holomorphic mappings*, Math. Ann. 279 (1987), 239–252.

[3] G. Herbort, *Estimation of the Carathéodory distance on pseudoconvex domains of finite type, whose boundary has a Levi form of corank at most one*, Ann. Polon. Math. 109 (2013), 209–260.

[4] M. Jarnicki, N. Nikolov, *Behavior of the Carathéodory metric near strictly convex boundary points*, Univ. Iag. Acta Math. XL (2002), 7–12.

[5] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis*, de Gruyter Exp. Math. 9, de Gruyter, Berlin.

[6] N. Nikolov, *Estimates of invariant distances on "convex" domains*, Ann. Mat. Pura Appl. 193 (2014), 1595–1605.

[7] N. Nikolov, *Comparison of invariant functions on strongly pseudoconvex domains*, J. Math. Anal. Appl. 421 (2015), 180–185.

[8] N. Nikolov, P. Pflug, P. J. Thomas, *Upper bound for the Lempert function of smooth domains*, Math. Z. 266 (2010), 425–430.

[9] S. Venturini, *Comparison between the Kobayashi and the Carathéodory distances on strongly pseudoconvex bounded domains in $\mathbb{C}^n$*, Proc. Am. Math. Soc. 107 (1989), 725–730.

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev 8, 1113 Sofia, Bulgaria

Faculty of Information Sciences, State University of Library Studies and Information Technologies, Shipchenski prohod 69A, 1574 Sofia, Bulgaria

E-mail address: nik@math.bas.bg

Institute of Mathematics, Faculty of Mathematics and Computer Science,, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

E-mail address: maria.trybula@im.uj.edu.pl