INVARIANT METRICS ON THE SYMMETRIZED BIDISC

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Abstract. We find an explicit formula of the Bergman and the Carathéodory metrics for the symmetrized bidisc $\mathbb{G}_2$. We also show that Carathéodory metric for $\mathbb{G}_2$ is not differentiable.

1. Introduction

We study invariant metrics on the symmetrized bidisc. The main part is devoted to the Bergman metric. Using the orthonormal basis for the space $\mathbf{L}_h^2(\mathbb{G}_2)$ and the Jacobi-Trudy identities we calculate this metric on $\mathbb{G}_2$. Approach which we present might be applicable to any domain and in any dimension. The only problem lies in finding the sum of some series. In fact, this method works for the Bergman kernel function and sectional curvature, too. In the last part we study Carathéodory metric on this domain.

The symmetrized bidisc is the first known example of a domain not biholomorphic to a convex one for whose the Lempert theorem holds, s.e. [5]). Moreover, it plays also an important role in solving Pick-Nevanlinna Interpolation Problem in dimension $n = 2$ (see e.g. [1]). Thus it is worth to understand better its geometric properties.

There are few examples of domains for whose the Bergman metric is known: Euclidean ball, minimal ball, polydisc and some special elipsoids (see e.g [7]), and this is an additional reason which encourages us to find $\beta_{\mathbb{G}_2}$.

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2. The Bergman metric for $\mathbb{G}_2$

Let $s : \mathbb{C}^2 \to \mathbb{C}^2$ be the symmetrization function given by the formula

$$s(z_1, z_2) = (z_1 + z_2, z_1 z_2) =: \big(s_1(z_1, z_2), s_2(z_1, z_2)\big), \quad z_1, z_2 \in \mathbb{C}.$$

Recall that the map $s|_{\mathbb{D}^2} : \mathbb{D}^2 \to s(\mathbb{D}^2) =: \mathbb{G}_2$ is a proper holomorphic one (see e.g. [11]) and its image $\mathbb{G}_2$ is called *the symmetrized bidisc.* We put $s_0(z_1, z_2) = 1,\ s_k(z_1, z_2) = 0$ for $k \neq 0,\ 1,\ 2$.

Let D be a bounded domain in $\mathbb{C}^2$. Denote by K_D and β_D the Bergman kernel function and the Bergman metric (for basic properties see [7]), respectively:

$$K_D(z, w) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(w)},$$

$$\beta_D^2(z; X) = \sum_{1 \leq j,k \leq 2} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log K_D(z, z) X_j \overline{X}_k, \quad \text{for } z,\ w \in D,\ X \in \mathbb{C}^2,$$

where $\{\varphi_j\}$ is an orthonormal basis for

$$\mathbf{L}_h^2(D) = \{f \,\epsilon\, \mathcal{O}(D) : \int_D |f|^2 dV < \infty\}.$$

It turns out that the above definition of the Bergman metric is not good tool for explicit calculation. Therefore, we use another attitude, but which is equivalent to the above one. So, an alternative description is as follows

$$(1) \quad \beta_D(z; X) = \frac{1}{\sqrt{K_D(z, z)}} \sup\{|f'(z)X| : f \,\epsilon\, \mathcal{O}(D), f(z) = 0, \|f\|_{\mathbf{L}_h^2(D)} \leq 1\}.$$

Also the Bergman kernel function on the diagonal might be represented as a solution of some extremal problem, that is

$$K_D(z, z) = \sup\{|f(z)|^2 : \|f\|_{\mathbf{L}_h^2(D)} \leq 1\}, \quad z \in D.$$

One of the most important property of the Bergman metric is that it is invariant under biholomorphic mappings. Recall that the group of automorphisms of the symmetrized bidisc $\mathrm{Aut}(\mathbb{G}_2)$ (see [8]) consists of mappings of the form

$$H(z_1 + z_2, z_1 z_2) = (s_1(h(z_1), h(z_2)), s_2(h(z_1), h(z_2))), \quad z_1, z_2 \,\epsilon\, \mathbb{D},$$

where $h \,\epsilon\, \mathrm{Aut}(\mathbb{D})$. Therefore, it is enough to compute the Bergman metric in the symmetrized bidisc at points of the form $(0, s_2)$ with $s_2 \in [0, 1)$.

A finite sequence $p = (p_1, \ldots, p_n)$ of decreasing (not necessarily strictly) non-negative integers we call *partition* (n is the length of the partition). By $[n]$ denote the set of all partitions of size n. If $\delta := (n-1, \ldots, 0)$, then $[[n]] := [n] + \delta$. We shall define constant

$$c_p^2 = \frac{p_1(p_2+1)}{\pi^2},$$

and Schur polynomial

$$S_p(z) = \frac{a_p(z)}{a_\delta(z)},\ z \,\epsilon\, \mathbb{D}^2,$$

where $a_p(z) := z_1^{p_1} z_2^{p_2} - z_1^{p_2} z_2^{p_1}$ for any $p \,\epsilon\, [[2]]$ (It is a very special case of more general situation - see e.g. [6]). Elementary calculation shows that S_p is actually polynomial. We additionally define $S_p = 0$, if $p \in \mathbb{Z}^2 \setminus [[2]]$. From [9] we know that the complete orthonormal system for $\mathbf{L}_h^2(\mathbb{G}_2)$ is

$$\{e_p = c_p S_p : p \,\epsilon\, [[2]]\}.$$

There are some relations between Schur polynomials and elementary symmetric functions, called *the Jacobi-Trudy identities* (see [6]). But to understand them we need the notion of conjugate partition. If $\lambda = (\lambda_1, \ldots, \lambda_n) \,\epsilon\, [n]$, then *conjugate partition* to λ is a partition $\mu = (\mu_1, \ldots, \mu_{\lambda_1})$ (denoted by λ') such that $\mu_k = \#\{j : \lambda_j \geq k\}$. The Jacobi-Trudy identities are described as following

$$S_{p+(1,0)} = \det[s_{\mu_l+k-l}]_{k,l=1,\ldots,p_1-1}.$$

They imply

Lemma 1.

$$S_{(k+m,m)+\delta}(s_1, s_2) = s_2^m \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^l \binom{k-l}{l} s_1^{k-2l} s_2^l,\ \text{for } k,\, m \geqslant 0,$$

where the symbol $\lfloor\cdot\rfloor$ denotes the greatest integer function.

Let A be any non empty set. Consider

$$l^2(A) := \{(x_a)_{a \epsilon A} : x_a \,\epsilon\, \mathbb{C},\ \sum_{a \epsilon A} |x_a|^2 < \infty\}.$$

It is the Hilbert space with the standard scalar product

$$\langle x, y \rangle := \sum_{a \epsilon A} x_a \overline{y_a}.$$

If $B \subset A$, then the space $l^2(B)$ is naturally embeded in $l^2(A)$.

Theorem 1. *For* $s_2 \,\epsilon\, [0,1)$

$$\beta_{\mathbb{G}_2}\Big((0,s_2);(X_1,X_2)\Big) = \sqrt{B_1|X|_1^2 + B_2|X|_2^2}\,,$$

where $B_1 = \frac{4x^2+10x+2}{(1-x)^2(3x+1)},\ B_2 = \frac{39x^2+18x+7}{(1-x)^2(3x+1)^2}$ *with* $x = s_2^2$.

Proof. Recall that

$$\beta_{\mathbb{G}_2}\Big((0,s_2);X\Big) = \frac{1}{\sqrt{K_{\mathbb{G}_2}((0,s_2),(0,s_2))}} \times$$
$$\sup\{|f'(0,s_2)(X)| : f \,\epsilon\, \mathbf{L}_h^2(\mathbb{G}_2), f(0,s_2)=0, \|f\|_{\mathbf{L}_h^2(\mathbb{G}_2)} \leqslant 1\}.$$

Denote by c, x_0 vectors:

$$x_0 = \left\{\Big((-1)^n\sqrt{(\frac{(k+2n+1)(k+1)}{\pi^2})}\ s_2^{n+k}\Big)_{(k+2n+1,k)}\right\}_{k,\,n\geqslant 0},$$

and

$$c = \left\{\Big((-1)^n(n{+}k)\sqrt{\frac{(k+2n+1)(k+1)}{\pi^2}}\ s_2^{n+k-1}\overline{X_2}\Big)_{(k+2n+1,k)}\right\}_{k,\,n\geqslant 0}$$
$$\cup\left\{\Big((-1)^n(n+1)\sqrt{\frac{(k+2n+2)(k+1)}{\pi^2}}\ s_2^{n+k}\overline{X_1}\Big)_{(k+2n+2,k)}\right\}_{k,\,n\geqslant 0}.$$

Vector c induces bounded operator

$$\Lambda : l^2([[2]]) \to \mathbb{C},$$
$$\Lambda(z) = \langle z, c\rangle\,.$$

Fix $f = \sum_{p\,\epsilon\,[[2]]}\alpha_p c_p S_p$ from $\mathbf{L}_h^2(\mathbb{G}_2)$. Let $\alpha = \{\alpha_p\}_{p\in[[2]]}$. Notice that

$$f'(0,s_2)X = \Lambda(\alpha).$$

Consequently, the supremum which appears in formula for $\beta_{\mathbb{G}_2}$ is equal to

$$\sqrt{K_{\mathbb{G}_2}((0,s_2),(0,s_2)}\ \beta_{\mathbb{G}_2}((0,s_2);X) = \|\Lambda|_{\{x_0\}^\perp}\|$$

(norm of the operator Λ restricted to $\{x_0\}^\perp$). Before we write explicite formula for that supremum we state

Lemma 2. *Let* $\Lambda : H \to \mathbb{C}$ *be any bounded linear functional on a Hilbert space* H *and let* $H = \bigoplus_{1\le j\le n} H_j$ *be a direct product of pairwise orthogonal subspace* H_j *of* H*, then*

$$\|\Lambda\|^2 = \sum_{1\le j\le n}\|\Lambda|_{H_j}\|^2.$$

Moreover, the statement remains true for countably many orthogonal subspaces.

Proof. It is a consequnce of the Riesz representation theorem and the Pythagorean theorem. ☐

So

$$\|\Lambda|_{\{x_0\}^\perp}\|^2 = \|\Lambda\|^2 - \|\Lambda|_{\mathrm{span}\{x_0\}}\|^2 = \langle c, c\rangle - \frac{\big|\langle c, x_0\rangle\big|^2}{\langle x_0, x_0\rangle}.$$

We put $x = s_2^2$. To finish the proof it is enough to find the remaining scalar products:

$$\langle x_0, x_0\rangle = K_{\mathbb{G}_2}((0,s_2),(0,s_2)) = \frac{3x+1}{\pi^2(1-x)^4},$$

$$\langle c, c\rangle = \frac{4x^2+10x+2}{\pi^2(1-x)^6}\,|X_1|^2 + \frac{27x^2+46x+7}{\pi^2(1-x)^6}\,|X_2|^2,$$

$$|\langle c, x_0|^2 = \frac{(9x+7)^2 x}{\pi^4(1-x)^{10}}\,|X_2|^2.$$

☐

3. The Carathéodory metric for the symmetrized bidisc

In [2] the authors computed the Carathéodory pseudodistance $c_{\mathbb{G}_2}$ for $\mathbb{G}_2$ for the origin, so one can easily find Carathéodory-Reiffen $\gamma_{\mathbb{G}_2}$ metric for $\mathbb{G}_2$ at origin (for definitions s.e. [7]). Recall, that if (s_1, s_2) is a point from $\mathbb{G}_2$ then (s.e. [2])

$$c_{\mathbb{G}_2}((0,0),(s_1,s_2)) = \frac{2|s_1 - s_2\overline{s}_1| + |s_1^2 - 4s_2|}{4 - |s_1|^2}.$$

Consequently,

$$\gamma_{\mathbb{G}_2}((0,0);(X_1,X_2)) = \frac{|X_1| + 2|X_2|}{2}, \tag{2}$$

where $(X_1, X_2) \in \mathbb{C}^2$.

A formula for $\gamma_{\mathbb{G}_2}$ was derived indepently by Costara and Agler, Young (s.e. [5]). Recall

$$\begin{aligned} &\gamma_{\mathbb{G}_2}\big((0,p);X\big) \\ &\quad = \max\Big\{\gamma_{\mathbb{D}}\big(f_\omega(0,p); f_\omega'(0,p)(X)\big) : f_\omega(x,y) = \frac{2\omega y - x}{2-\omega x},\ \omega \in \mathbb{T}\Big\}. \end{aligned} \tag{3}$$

Note, that

$$\gamma_{\mathbb{D}}\big(f_\omega(0,p); f_\omega'(0,p)(X)\big) = \frac{\big|\omega p X_1 + 2X_2 - \overline{\omega} X_1\big|}{2(1-p^2)}, \tag{4}$$

so $\gamma_{\mathbb{G}_2}((0,p);(X_1,X_2)) = \gamma_{\mathbb{G}_2}((0,p);(X_1,\overline{X}_2)) = \gamma_{\mathbb{G}_2}((0,p);(X_1,-X_2))$ for $X_1 \geq 0,\ 1 > p \geq 0$. Thus, if $X_2 = r_2 e^{i\phi}$ with $r_2 \geq 0$, we can assume $\phi \in [0, \frac{\pi}{2}]$.

Now, take a, b positive real numbers and consider equation

$$H(\lambda) = \lambda^4 - \lambda^2(2 + a^2 + b^2) + 2\lambda(a^2 - b^2) + (1 - a^2 - b^2) = 0. \tag{5}$$

Note, (5) has only one solution in $(-\infty, -1)$. Indeed, let us define $G(\lambda) = \frac{a^2}{(\lambda+1)^2} + \frac{b^2}{(\lambda-1)^2}$, and notice that if $\lambda \in (-\infty, -1)$, then $G(\lambda) = 1$ iff $H(\lambda) = 0$.

To formulate the next lemma we shall need some auxiliary constants

$$a = \frac{r_2 \sin\phi\,(p+1)}{pr_1}, \quad b = \frac{r_2 \cos\phi\,(1-p)}{pr_1}, \tag{6}$$

where $r_1 > 0,\ r_2 \geq 0,\ 1 > p > 0,\ \frac{\pi}{2} \geq \phi \geq 0$.

Lemma 3. *Let $p \in (0,1)$ and $X = (X_1, X_2) = (r_1, r_2 e^{i\phi}) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}$. For $r_1 r_2 \neq 0$ let a, b be the constants given by (6), and let λ be the only root in $(-\infty, -1)$ of the polynomial (5). Then*

$$\gamma_{\mathbb{G}_2}\big((0,p);X\big) = \begin{cases} \dfrac{\sqrt{(p+1)^2|X_1|^2 + (4 + \frac{(1-p)^2}{p})|X_2|^2}}{2(1-p^2)} & \text{if } pr_1r_2 \neq 0,\ \sin\phi = 0,\ b \leqslant 2, \\[2ex] \dfrac{\sqrt{[1 + p^2 - 2p(2\lambda+1) + \frac{4pb^2}{(1-\lambda)^2}]|X_1|^2 + 4|X_2|^2}}{2(1-p^2)} & \text{if } pr_1r_2 \neq 0,\ \sin\phi \neq 0, \text{ or} \\ & \text{if } pr_1r_2 \neq 0,\ \sin\phi = 0,\ b \geqslant 2. \end{cases}$$

Remark 1. *Cases not covered by the Lemma 3 can be achieved by considering the relevant limits (recall $\Gamma_{\mathbb{G}_2}$ is locally Lipschitz - see e.g.* [7]*).*

Proof. The square of numerator in (4) can be written as follows ($\omega = e^{i\theta}$)

$$F(\theta) = r_1^2(p^2+1) + 4r_2^2 + 2pr_1^2\Big[\big(\sin\theta + a\big)^2 - \big(\cos\theta + b\big)^2 - a^2 + b^2\Big].$$

To localize the global maximum of the function $f(x,y) = (x+a)^2 - (y+b)^2$ on a set $x^2 + y^2 = 1$ it is enough to apply Lagrange multipliers. □

Remark 2. *From Lemma 3 we may deduce that the Carathéodory metric is not differentiable. Indeed, the differentiability is lost at points $((0,p);(1,r_2))$, where $p \in (0,1)$ and r_2 is the positive real number such* *that b in (6) is 2. For that it is enough to consider the limits:*

$$\lim_{\mathbb{R}\ni X_2\to r_2^+}\frac{\gamma_{\mathbb{G}_2}((0,p);(1,X_2))-\gamma_{\mathbb{G}_2}((0,p);(1,r_2))}{X_2-r_2}=Cp(\frac{2(1-p)}{p}+\frac{4}{1-p}),$$

and

$$\lim_{\mathbb{R}\ni X_2\to r_2^-}\frac{\gamma_{\mathbb{G}_2}((0,p);(1,X_2))-\gamma_{\mathbb{G}_2}((0,p);(1,r_2))}{X_2-r_2}=Cp(\frac{1-p}{p}+\frac{4}{1-p}),$$

where $C=\frac{1}{2(1-p^2)^2\gamma_{\mathbb{G}_2}((0,p);(1,r_2))}$.

Remark 3. *One might check that* $\beta_{\mathbb{G}_2}/\gamma_{\mathbb{G}_2}\geq 1,41421$*. It seems that this quotient is greater or equal to* $\sqrt{2}$*. Since* λ *is given as a solution of (5), it is much more difficult to get any reasonably estimate from above.*

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