Rigorous numerics for PDEs with indefinite tail: existence of a periodic solution of the Boussinesq equation with time-dependent forcing[*]

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Abstract. We consider the Boussinesq PDE perturbed by a time-dependent forcing. Even though there is no smoothing effect for arbitrary smooth initial data, we are able to apply the method of self-consistent bounds to deduce the existence of smooth classical periodic solutions in the vicinity of 0. The proof is non-perturbative and relies on construction of periodic isolating segments in the Galerkin projections.

Keywords: Boussinesq equation, ill-posed PDEs, periodic solutions, isolating segments.

1. Introduction

In recent years pioneering contributions have been made in the field of rigorous computer-assisted results for dynamics of dissipative PDEs [2,3,5,6,8,9,16,17,19,20,21]. The methods exploit the smoothing property of the system to apply either topological or functional-analytic tools. However, little attention has been paid to apply these methods to other types of evolution PDEs, such as the ones with tail of saddle type. In such problems we need to deal with an infinite number of strongly repelling and

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strongly attracting directions. From the point of view of topological methods this situation is just as good as the dissipative one – for example Theorem 2.3 in [16] is formulated in a way that is readily applicable to finding equilibria of these systems. Our goal in this paper is to take this approach one step further and use a topological tool of periodic isolating segments to prove the existence of periodic in time solutions in a nonautonomously perturbed equation of such type.

Our example will be the Boussinesq equation [4] but without much difficulty the methods can be applied to produce similar results in other systems with an indefinite tail of saddle type.

1.1. The forced Boussinesq equation

We consider the following second order nonlinear equation perturbed by a time-dependent forcing term:

$$u_{tt} = u_{xx} + \beta u_{xxxx} + \sigma(u^2)_{xx} + \epsilon f(t, x) \,. \tag{1}$$

On u and f we impose periodic and even boundary conditions and a zero-average condition in x:

$$u(t, x + 2\pi) = u(t, x) \,, \tag{2}$$

$$u(t, -x) = u(t, x) \,, \tag{3}$$

$$\int_0^{2\pi} u(t, x) dx = 0 \,, \tag{4}$$

$$f(t, x + 2\pi) = f(t, x) \,, \tag{5}$$

$$f(t, -x) = f(t, x) \,, \tag{6}$$

$$\int_0^{2\pi} f(t, x) dx = 0 \,. \tag{7}$$

For $\beta > 0$ the unperturbed equation

$$u_{tt} = u_{xx} + \beta u_{xxxx} + \sigma(u^2)_{xx} \tag{8}$$

is the "bad" Boussinesq equation and was derived by Boussinesq [4] as a model for shallow water waves. The equation is famous for its ill-posedness. Indeed, when looking at its linear part

$$u_{tt} = u_{xx} + \beta u_{xxxx} \tag{9}$$

one can observe a rapid growth in high Fourier modes for almost all initial data, hence a consequent loss of regularity of the solution. This is a significant complication in the numerical analysis of (8), since slightest perturbations of the initial problem can produce a totally different behaviour at output. Because of that, regularized versions of the equation were considered in numerical studies [11]. Solutions to the equation (8) were also obtained analytically [10] and by the inverse scattering method [15]. Our approach is different; we analyze the direction of the vector field

Tab 1.: Bounds on ϵ and on the norms of periodic solutions and their time derivatives for $f \in \mathcal{F}_\tau^A$

β	ϵ	$\|u(t,\cdot)\|_{L^2} \leq$	$\|u(t,\cdot)\|_{C^0} \leq$	$\|u_t(t,\cdot)\|_{L^2} \leq$	$\|u_t(t,\cdot)\|_{C^0} \leq$
1.5	$[-0.05, 0.05]$	0.28862115	0.12440683	0.24610779	0.2143523
1.75	$[-0.1, 0.1]$	0.41504192	0.1820825	0.39340084	0.42461205
2.5	$[-0.3, 0.3]$	0.84724825	0.38676747	0.96839709	1.4795576

Tab 2.: Bounds on ϵ and on the norms of periodic solutions and their time derivatives for $f \in \mathcal{F}_\tau^B$

β	ϵ	$\|u(t,\cdot)\|_{L^2} \leq$	$\|u(t,\cdot)\|_{C^0} \leq$	$\|u_t(t,\cdot)\|_{L^2} \leq$	$\|u_t(t,\cdot)\|_{C^0} \leq$
1.5	$[-0.05, 0.05]$	0.29831987	0.13194161	0.25703095	0.24099758
1.75	$[-0.1, 0.1]$	0.43198386	0.19524766	0.41478653	0.47720834
2.5	$[-0.3, 0.3]$	0.88825406	0.41784158	1.0309512	1.637095

on certain subsets of the phase space, and by a topological method we deduce the existence of smooth, periodic solutions.

Here is an example result illustrating our method. We define families of functions

$$\mathcal{F}_\tau^A = \{f : \ f(t,x) = 2f_1(t)\cos x,$$
$$f_1 \text{ continuous and } \tau\text{-periodic}, \ |f_1(t)| \leq 1 \ \forall t\},$$

$$\mathcal{F}_\tau^B = \{f : \ f(t,x) = 2\sum_{k=1}^{4} f_k(t)\cos kx, \tag{10}$$
$$f_k \text{ continuous and } \tau\text{-periodic}, \ |f_k(t)| \leq 1 \ \forall k,t\}.$$

Then the following theorem holds.

THEOREM 1. *For $\sigma = 3$, for all $\tau > 0$ and all $f(t,x) \in \mathcal{F}_\tau^A \cup \mathcal{F}_\tau^B$, and for values of β and ϵ given in Tables 1 and 2 there exists a classical τ-periodic in time solution to* (1), *subject to conditions* (2), (3) *and* (4). *The solution exists in the vicinity of 0 and the bounds on its L^2 and C^0 norms and the norms of its time derivative are given in Tables 1 and 2.[1] The solution and its time derivative are C^4 and C^2 smooth in x, respectively.*

Observe, that 0 is a constant in time solution of the unperturbed system (8), hence the requested τ-periodic solution for $\varepsilon = 0$. Nevertheless, the method is not perturbative. We consider a perturbation problem only because it gives a convenient approximation of the periodic solution for $|\epsilon| \neq 0$ small.

The proof is computer-assisted, that means certain inequalities contained in it are verified rigorously by a computer program in interval arithmetics. The program source code is available at [7]. From Table 3 and equation (26) one can also extract the exact bounds on the Fourier coefficients of the solutions, which we do not give

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[1]By bounds on L^2 and C^0 norms of a function $u = u(t,x)$ we mean upper bounds on $\sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_{L^2}$ and $\sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_{C^0}$, respectively.

here. By following the steps of the proof it will become clear that we can easily produce results of the same type for:

- any parameters $\sigma \in \mathbb{R}$ and $\beta > 1$,

- any given smoothness $s > 5$,

- any forcing of the form $f(t, x) = \sum_{k=1}^{n} f_k(t) \cos kx$ where each f_k is continuous and τ-periodic.

The periodic solution will exist for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, ε_0 small enough, and we can attempt to verify an explicit range and obtain a bound for the norm with help of the program.

Let us finish this section with listing some generalizations. Most of them can be adapted from the previous treatment of dissipative PDEs [5, 6, 21] and several would require only little effort to be introduced in this paper. However, we consider this exposition as a preview of the method. We tried to focus on the key matter, which is how to deal with the linear instability of the high Fourier modes in a simplest scenario.

1. We could take β from the range $(0, 1]$. Then, the linearized equation 9 possesses purely imaginary eigenvalues in its low modes. This would involve conducting a finite-dimensional analysis of the higher order terms of the low modes.

2. We could allow non-zero averages, non-even functions or periodic solutions obtained in the proximity of non-zero equilibria of the unperturbed system.

3. For the forcing term it would have been enough to assume a sufficiently fast decay in the high Fourier terms.

4. We could consider non-periodic forcings and attempt to prove existence of (not necessarily periodic) solutions that exist for all $t \in \mathbb{R}$, by techniques from [6].

The apparently difficult problems are

- proving the existence of periodic solutions which are not obtained as perturbations of stationary points,

- proving dynamics more complicated than a periodic orbit (e.g. chaotic dynamics),

- proving the existence of periodic orbits in autonomous ill-posed systems.

We think that to efficiently treat these cases within the framework of self-consistent bounds we would need a rigorous integration procedure, akin to the rigorous integration of dissipative PDEs [5, 19, 20]. Obviously, the ill-posedness is a significant issue and it seems that the integration should be combined with an automatic segment placement in the expanding coordinates. We are currently looking into the feasability of this approach.

This paper is organized as follows. In Section 2 we invoke the general method of self-consistent bounds and a result which states that a sequence of solutions for the Galerkin projections converges to a solution of the PDE. In Section 3 we present a

result of Srzednicki [12] stating the conditions under which non-autonomous time-periodic ordinary differential equations – in our case the Galerkin projections – have periodic solutions. In Section 4 we apply these tools to the Boussinesq equation (1) and prove Theorem 1.

2. The general method of self-consistent bounds

In this section we recall the general method of self-consistent bounds, as introduced in the series of papers [19,20,21]. We follow the exposition given in [6] for time-dependent systems.

Let $J \subset \mathbb{R}$ be a (possibly unbounded) interval. We consider a nonautonomous evolution equation on a real Hilbert space H (L^2 in our case) of the form

$$\frac{da}{dt} = F(t, a).$$

(11)

We assume that the set of x such that $F(t, x)$ is defined for every $t \in J$ is dense in H and we denote it by $\tilde{H}$. By a solution of (11) we mean a function $a : J' \to \tilde{H}$, such that J' is a subinterval of J, u is differentiable and (11) is satisfied for all $t \in J'$.

Let $I \subset \mathbb{Z}^d$ and let $H_k \subset H$ be a sequence of subspaces with $\dim H_k \leq d_1 < \infty$, such that

$$H = \overline{\bigoplus_{k \in I} H_k}$$

(12)

and H_k's are pairwise orthogonal. We will denote the orthogonal projection onto H_k by A_k and write

$$a_k := A_k a.$$

(13)

From (12) it follows that $a = \sum_{k \in I} a_k$.

From now on we will fix some (arbitrary) norm $|\cdot|$ on $\mathbb{Z}^d$. For $n > 0$ we set

$$X_n := \bigoplus_{k \in I, |k| \leq n} H_k,$$

$$Y_n := X_n^\perp.$$

(14)

We will denote the orthogonal projections onto X_n and Y_n by $P_n : H \to X_n$ and $Q_n : H \to Y_n$.

DEFINITION 2. *Let $J \subset \mathbb{R}$ be an interval. We say that $F : J \times H \supset \operatorname{dom} F \to H$ is admissible, if the following conditions hold for each $i \in \mathbb{Z}^d$ such that $\dim X_i > 0$:*

- $J \times X_i \subset \operatorname{dom} F$,

- $(P_i \circ F)|_{J \times X_i} : J \times X_i \to X_i$ *is a C^1 function.*

DEFINITION 3. *Let $F : J \times \tilde{H} \to H$ be admissible. The ordinary differential equation*

$$\frac{dp}{dt} = P_n F(t, p), \quad p \in X_n, \tag{15}$$

will be called the n-th Galerkin projection of (11).

DEFINITION 4. *Assume $F : J \times \tilde{H} \to H$ is an admissible function. Let $m, M \in \mathbb{N}$ with $m \leq M$. A compact set consisting of $W \subset X_m$ and a sequence of compact sets $\{B_k\}_{k \in I, |k| > m}$ such that $B_k \subset H_k$ form self-consistent bounds if the following conditions are satisfied:*

C1 *For $|k| > M$, $k \in I$ it holds that $0 \in B_k$,*

C2 *Let $|k| > m$ and $\hat{a}_k := \max_{a \in B_k} ||a||$. Then $\sum_{|k|>m, k \in I} \hat{a}_k^2 < \infty$. In particular we have*

$$W \oplus \prod_{|k|>m} B_k \subset H. \tag{16}$$

C3 *The function $(t, u) \to F(t, u)$ is continuous on $J \times W \oplus \prod_{|k|>m} B_k \subset \mathbb{R} \times H$. Moreover, if we define $\hat{f}_k := \sup_{(t,u) \in J \times W \oplus \prod_{k \in I, |k|>m} B_k} ||A_k F(t, u)||$, then $\sum_{|k|>m, k \in I} \hat{f}_k^2 < \infty$.*

Given self-consistent bounds formed by W and $\{B_k\}_{k \in I, |k|>m}$, by T (*the tail*) we will denote the set

$$T := \prod_{k \in I, |k|>m} B_k \subset Y_m. \tag{17}$$

The following theorem is a straightforward adaptation of Lemma 5 from [19] (see also Section 4 in [18]) to a nonautonomous setting.

THEOREM 5. *Let W and $\{B_k\}_{k \in I, |k|>m}$ form self-consistent bounds and let $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ be a sequence, such that $\lim_{k \to \infty} n_k = \infty$. Assume that for all $k > 0$ there exists a solution $x_k : [t_1, t_2] \to W \oplus T$ of*

$$\frac{dp}{dt} = P_{n_k} (F(t, p(t))), \quad p(t) \in X_{n_k}. \tag{18}$$

Then there exists a convergent subsequence $\lim_{l \to \infty} p_{k_l} = p^*$, where $p^* : [t_1, t_2] \to W \oplus T$ is a solution of (11). *Moreover, the convergence is uniform with respect to t on $[t_1, t_2]$.*

It turns out that it is fairly simple to find self-consistent bounds. In the treatment of evolution PDEs such as Kuramoto-Sivashinsky or Navier-Stokes it is enough to take tails of the form $B_k = \{a \in H_k : ||a|| \leq C/|k|^s\}$ for s large enough. This will also be the case in our study of the Boussinesq equation.

3. Periodic isolating segments

The purpose of this section is to recall a result of Srzednicki [12] on the existence of periodic orbits in non-autonomous time-periodic ODEs. For the Boussinesq equation this theorem will be used to treat each of the Galerkin projections of the system.

We consider an ODE

$$\dot{x} = g(t, x), \tag{19}$$

where $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 and τ-periodic in t. Let $L^i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$ be C^1 functions and let $r \in \{0, \ldots, k\}$ be fixed. We define sets

$$
\begin{aligned}
S_0 &:= \{x \in \mathbb{R}^n : L^i(x) \leq 0 \ \forall i = 1, \ldots, k\}, \\
S_0^- &:= \{x \in \mathbb{R}^n : \exists i \in 1, \ldots, r : L^i(x) = 0\},
\end{aligned}
\tag{20}
$$

and put $S := [0, \tau] \times S_0$, $S^- := [0, \tau] \times S_0^-$.

DEFINITION 6. *We call S a periodic isolating segment*[2] *over $[0, \tau]$ and S^- its exit set iff the following conditions hold*

(S1) $g(t, x) \cdot \nabla L^i(x) > 0$ *for $t \in [0, \tau]$, $i \in \{1, \ldots, r\}$ and $x \in S_0 : L^i(x) = 0$.*

(S2) $g(t, x) \cdot \nabla L^i(x) < 0$ *for $t \in [0, \tau]$, $i \in \{r+1, \ldots, k\}$ and $x \in S_0 : L^i(x) = 0$.*

THEOREM 7 (Theorem 2 in [12]). *Let S be a periodic isolating segment over $[0, \tau]$ for g defined as above. If S and S^- are compact absolute neighborhood retracts and the difference of their Euler characteristics $\chi(S_0) - \chi(S_0^-)$ is non-zero, then there exists a point $x_0 \in \mathrm{int}\, S_0$ such that the solution $x(t)$ of (19) satisfying $x(0) = x_0$ is τ-periodic in t and $x(t) \in \mathrm{int}\, S_0$ for all $t \in \mathbb{R}$.*

In what is below we denote the i-th coordinate of a vector $x \in \mathbb{R}^n$ by x_i.

COROLLARY 8. *Let $S_0 := [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $g = (g_1, \ldots, g_n)$ be defined as above. Suppose that for each i it holds that*

$$g_i(t, x) g_i(t, y) < 0 \tag{21}$$

for all $x, y \in S_0 : x_i = a_i$, $y_i = b_i$ and for all $t \in [0, \tau]$. Then there exists a τ-periodic in t solution of (19) with values contained in $\mathrm{int}\, S_0$.

Proof. After rearranging the coordinates we can assume that

$$
\begin{aligned}
g_i(t, x) &> 0, \quad \text{for } x \in S_0 : \ x_i \in \{b_1, \ldots, b_r, a_{r+1}, \ldots, a_n\}, \\
g_i(t, x) &< 0, \quad \text{for } x \in S_0 : \ x_i \in \{a_1, \ldots, a_r, b_{r+1}, \ldots, b_n\}.
\end{aligned}
\tag{22}
$$

for all $t \in [0, \tau]$ and some $r \in \{1, \ldots, n\}$.

Let $L^i(x) := (x_i - a_i)(x_i - b_i)$, $i = 1, \ldots, n$. Then S_0 is given by the sequence $\{L^i\}$ and

$$\nabla L^i(x) = 2x_i - a_i - b_i. \tag{23}$$

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[2]The original paper [12] uses the notion of (p, q)-blocks, periodic isolating segments are introduced in later studies [13, 14] as a more general tool.

Now conditions (S1) and (S2) immediately follow from (22), hence $S = [0, \tau] \times S_0$ forms a periodic isolating segment. Since S_0^- consists of r opposite faces, we have $\chi(S_0) - \chi(S_0^-) = (-1)^r$ and the assertion of Theorem 7 follows. $\qquad \square$

Later on we will refer to condition (21) as the *isolation conditions* or the *isolation inequalities*.

4. Application to the Boussinesq PDE

4.1. The Boussinesq equation in the Fourier basis

Our goal in this section is to express our PDE in coordinates suitable for application of Corollary 8 to the Galerkin projections. For that purpose we express the problem in the Fourier basis and diagonalize its linear part.

By formally substituting $u(t, x) = \sum_{k \in \mathbb{Z}} u_k(t) e^{ikx}$ into the Boussinesq equation (1) we obtain an infinite ladder of second order equations

$$\ddot{u}_k = k^2 (\beta k^2 - 1) u_k - \sigma k^2 \sum_{k_1 \in \mathbb{Z}} u_{k_1} u_{k-k_1} + f_k(t), \ k \in \mathbb{Z}. \tag{24}$$

Since u is real and even in x, we have $u_k = u_{-k}$ and $u_k \in \mathbb{R}$ for all $k \in \mathbb{Z}$. Moreover, from (4) we have $u_0 = 0$. After these substitutions and rewriting the system as a first order system we obtain the following equations

$$\dot{u}_k = v_k \,,$$

$$\dot{v}_k = k^2 (\beta k^2 - 1) u_k - 2\sigma k^2 \sum_{k_1 \geq 1} u_{k_1 + k} u_{k_1} - \sigma k^2 \sum_{k_1 = 1}^{k-1} u_{k_1} u_{k-k_1} + \epsilon f_k(t), \ k \in \mathbb{N}^+ \,.$$
$$\tag{25}$$

As one can see, the linear part of (25) is already in a block-diagonal form. All we need is to diagonalize each of the blocks. From now on we assume that $\beta > 1$. After a simple calculation we see that the eigenvalues of the linear part of (25) are $\pm\sqrt{k^2(\beta k^2 - 1)}$ with eigenvectors $\left[1, \pm\sqrt{k^2(\beta k^2 - 1)}\right]^T$, respectively. We introduce the variables u_k^+ and u_k^- such that

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \sqrt{k^2(\beta k^2 - 1)} & -\sqrt{k^2(\beta k^2 - 1)} \end{bmatrix} \begin{bmatrix} u_k^+ \\ u_k^- \end{bmatrix}. \tag{26}$$

We have

$$\begin{bmatrix} u_k^+ \\ u_k^- \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{k^2(\beta k^2 - 1)}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{k^2(\beta k^2 - 1)}} \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix}, \tag{27}$$

and our equations become

$$
\dot{u}_k^+ = \sqrt{k^2(\beta k^2 - 1)}u_k^+ + \frac{\sigma k^2 N_k(u) + \epsilon f_k(t)}{2\sqrt{k^2(\beta k^2 - 1)}},
$$
$$
\dot{u}_k^- = -\sqrt{k^2(\beta k^2 - 1)}u_k^- - \frac{\sigma k^2 N_k(u) + \epsilon f_k(t)}{2\sqrt{k^2(\beta k^2 - 1)}}, \quad k \in \mathbb{N}^+,
$$

(28)

where

$$
N_k(u) := -2\sum_{k_1 \geq 1} u_{k_1+k} u_{k_1} - \sum_{k_1=1}^{k-1} u_{k_1} u_{k-k_1}.
$$

(29)

We remark that it is unprofitable to rewrite the convolutions in the new variables as we will eventually estimate these terms.

The n-th Galerkin projection of (28) is given by

$$
\dot{u}_k^+ = \sqrt{k^2(\beta k^2 - 1)}u_k^+ + \frac{\sigma k^2 N_{k,n}(u) + \epsilon f_k(t)}{2\sqrt{k^2(\beta k^2 - 1)}},
$$
$$
\dot{u}_k^- = -\sqrt{k^2(\beta k^2 - 1)}u_k^- - \frac{\sigma k^2 N_{k,n}(u) + \epsilon f_k(t)}{2\sqrt{k^2(\beta k^2 - 1)}}, \quad k = 1, \ldots, n,
$$

(30)

where

$$
N_{k,n}(u) := -2\sum_{k_1 \geq 1}^{n-k} u_{k_1+k} u_{k_1} - \sum_{k_1=1}^{k-1} u_{k_1} u_{k-k_1}.
$$

(31)

Our next step is to construct a sequence of isolating segments S^n for the Galerkin projections (30).

4.2. Construction of periodic isolating segments

We look for periodic isolating segments S^n of the form $S^n = [0, \tau] \times S_0^n$, where

$$
S_0^n = \prod_{k=1}^{M} [u_k^l, u_k^r]^2 \oplus \prod_{k=M+1}^{n} [-C/k^s, C/k^s]^2,
$$

(32)

i.e. the set of n-tuples of pairs $(u^+, u^-) = \{(u_k^-, u_k^+)\}_{k=1}^n$ such that $u_k^-, u_k^+ \in [u_k^l, u_k^r]$ for $k \in 1, \ldots, M$ and $\left|u_k^{\pm}\right| \leq C/k^s$. For now it is enough to take $C \in \mathbb{R}^+$ and $s \in \{2, 3, \ldots\}$, however later on we will assume that s is at least 6, to comply with condition **C3** from the definition of self-consistent bounds.

Observe that we would like to choose the values of C and s, as well as the first M intervals the same for each projection. Therefore we can say that our segments are a projection of an "infinite-dimensional segment" given by

$$
S_0^\infty := \prod_{k=1}^{M} [u_k^l, u_k^r]^2 \oplus \prod_{k=M+1}^{\infty} [-C/k^s, C/k^s]^2.
$$

(33)

The elements of S_0^∞ are sequences of pairs $(u^+, u^-) = \{(u_k^-, u_k^+)\}_{k=1}^\infty$. We denote them by the same symbols as elements of S_0^n but we will always make it clear to element of which set we are referring to.

We would like to choose u_k^l, u_k^r, C and s such that the linear part of (30) dominates the nonlinear terms and the isolation conditions (21) hold – at least for sufficiently high modes, for n large enough. The inequalities for the low modes we will treat one-by-one with aid of rigorous numerics.

We assume the bounds for u_k^+ to be the same as the ones for u_k^-. As we will see later, due to the symmetry of the equations (30), the isolation conditions for both u_k^+ and u_k^- are given by the same inequalities.

From estimates in [21][3] it follows that, for a set S_0^∞ of the form given above, there exists a constant $D \in \mathbb{R}^+$ such that

$$\sup\left\{|N_{k,n}(u)| : \ n \geq k > M, \ (u^+, u^-) \in S_0^n\right\} < \frac{D}{k^{s-1}}. \tag{34}$$

The value of D can be given by an explicit formula, but we postpone its evaluation to Subsection 4.3.

LEMMA 9. *Assume that for some $M \in \mathbb{N}^+$, we have $f_k = 0$, $k > M$ and*

$$C > \frac{\sigma D}{M+1} \cdot \frac{1}{2(\beta - (M+1)^{-2})}. \tag{35}$$

Consider the Galerkin projection (30) for $n \geq M$ and let S_0^n be given by (32). Then, for $M < k \leq n$ and $(u^+, u^-) \in S_0^n$ the following inequalities hold

$$\dot{u}_k^+ > 0 \quad if \quad u_k^+ = C/k^s, \tag{36}$$

$$\dot{u}_k^+ < 0 \quad if \quad u_k^+ = -C/k^s, \tag{37}$$

$$\dot{u}_k^- < 0 \quad if \quad u_k^- = C/k^s, \tag{38}$$

$$\dot{u}_k^- > 0 \quad if \quad u_k^- = -C/k^s. \tag{39}$$

Proof. We will prove (36) and (38). The proof of (37) and (39) follows by reversing the inequality signs. We want

$$\sqrt{k^2(\beta k^2 - 1)}u_k^\pm + \frac{\sigma k^2 N_{k,n}(u)}{2\sqrt{k^2(\beta k^2 - 1)}} > 0. \tag{40}$$

Since $u_k^\pm = C/k^s$, the above is equivalent to

$$\frac{C}{k^s} + \frac{\sigma N_{k,n}(u)}{2(\beta k^2 - 1)} > 0. \tag{41}$$

By the estimate (34) it is enough that

$$C > \frac{\sigma D k}{2(\beta k^2 - 1)} = \frac{\sigma D}{2k(\beta - k^{-2})} \tag{42}$$

and the right-hand side is at most $\frac{\sigma D}{M+1} \cdot \frac{1}{2(\beta - (M+1)^{-2})}$. $\qquad\square$

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[3]We note that the estimates from [20] improve the bound on (34) to $\tilde{D}/k^s$ for some $\tilde{D}$, but the one we use here is fine enough for our applications.

4.3. Estimates for the nonlinear terms

In this subsection we provide bounds for the nonlinear terms and compute D. In fact we will look for an estimate

$$N_{k,\max} < \frac{D}{k^{s-1}}, \quad k > M \tag{43}$$

for

$$N_{k,\max} := \sup\{|N_k(u)| : (u^+, u^-) \in S_0^\infty\}, \tag{44}$$

as it is an upper bound on the left-hand side of (34) for all $n, k : n \geq k$.

Let S_0^∞ be of the form as in (33) and $(u^+, u^-) \in S_0^\infty$. Recall that $u_k = u_k^+ + u_k^-$, hence

$$u_k \in [2u_k^l, 2u_k^r] \quad \text{for} \quad k \leq M,$$
$$|u_k| \leq \frac{2C}{k^s} \quad \text{for} \quad k > M. \tag{45}$$

The nonlinearity consists of terms given by an infinite sum and a finite sum

$$N_k(u) = -2IS(k) - FS(k), \tag{46}$$

where

$$IS(k) := \sum_{k_1 \geq 1} u_{k_1+k} u_{k_1}, \tag{47}$$

$$FS(k) := \sum_{k_1=1}^{k-1} u_{k_1} u_{k-k_1}. \tag{48}$$

These terms, arising from the nonlinearity in the Kuramoto-Sivasinsky equation, were estimated in [21] (cf. also [20], Section 8). Throughout the rest of this subsection we will denote by u_k the whole interval $[2u_k^l, 2u_k^r]$ and put $|u_k| := 2\max\{|u_k^l|, |u_k^r|\}$.

LEMMA 10 (Lemma 3.1 in [21]). *For $k \in \{1, \ldots, M\}$ we have*

$$IS(k) \subset \sum_{k_1=1}^{M-k} u_{k_1+k} u_{k_1} + 2C \sum_{k_1=M-k+1}^{M} \frac{|u_k|}{(k+k_1)^s} [-1, 1]$$
$$+ \frac{4C^2}{(k+M+1)^s (s-1) M^{s-1}} [-1, 1]. \tag{49}$$

LEMMA 11 (Lemma 3.5 in [21]). *For $k > 2M$ we have*

$$FS(k) \subset \frac{2C}{k^{s-1}} \left(\frac{2^{s+1}}{2M+1} \sum_{k_1=1}^{M} |u_k| + \frac{C2^{2s+1}}{(2M+1)^{s+1}} + \frac{C2^{s+1}}{(s-1)M^s} \right) [-1, 1]. \tag{50}$$

LEMMA 12 (Lemma 3.6 in [21]). *For $k > M$ we have*

$$IS(k) \subset \frac{2C}{k^{s-1}(M+1)} \left(\frac{2C}{(M+1)^{s-1}(s-1)} + \sum_{k=1}^{M} |u_k| \right) [-1, 1]. \tag{51}$$

Following [20] we give D_1, D_2 such that

$$|FS(k)| \leq \frac{D_1}{k^{s-1}}, \quad |IS(k)| \leq \frac{D_2}{k^{s-1}}, \quad k > M \tag{52}$$

and then set $D := D_1 + 2D_2$. Using Lemmas 11 and 12 we get the following formulas for D_1, D_2:

$$D_1(k \leq 2M) = \max\{k^{s-1}|FS(k)|, \ M < k \leq 2M\}, \tag{53}$$

$$D_1(k > 2M) = 2C \left(\frac{2^{s+1}}{2M+1} \sum_{k_1=1}^{M} |u_k| + \frac{C2^{2s+1}}{(2M+1)^{s+1}} + \frac{C2^{s+1}}{(s-1)M^s} \right), \tag{54}$$

$$D_1 = \max(D_1(k \leq 2M), D_1(k > 2M)), \tag{55}$$

$$D_2 = \frac{2C}{M+1} \left(\frac{2C}{(M+1)^{s-1}(s-1)} + \sum_{k=1}^{M} |u_k| \right). \tag{56}$$

4.4. Low mode isolation and a procedure for refining the bounds

We will now discuss the low mode isolation inequalities. Assume, that we found $M \in \mathbb{N}$ and segments S_0^n, $n > M$ such that the assumptions of Lemma 9 hold. To apply Corollary 8 to all Galerkin projections it is now enough to check

$$\dot{u}_k^+ > 0 \quad \text{if} \quad u_k^+ = u_k^r, \tag{57}$$

$$\dot{u}_k^+ < 0 \quad \text{if} \quad u_k^+ = u_k^l, \tag{58}$$

$$\dot{u}_k^- < 0 \quad \text{if} \quad u_k^- = u_k^r, \tag{59}$$

$$\dot{u}_k^- > 0 \quad \text{if} \quad u_k^- = u_k^l. \tag{60}$$

for $(u^-, u^+) \in S_0^n$ and $n > M$. It is enough to verify

$$u_k^r > -\frac{\sigma k^2 N_k(u) + \epsilon f_k(t)}{2k^2(\beta k^2 - 1)},$$
$$u_k^l < -\frac{\sigma k^2 N_k(u) + \epsilon f_k(t)}{2k^2(\beta k^2 - 1)}. \tag{61}$$

for all $u \in S_0^\infty$, $t \in [0, \tau]$ and $k = 1, \dots, M$. Recall that $N_k(u) = -2IS(k) - FS(k)$. The term $IS(k)$ is bounded by use of Lemma 10, while $FS(k)$ are finite sums which can be for example rigorously enclosed by use of interval arithmetics. Therefore we can compute an explicit bound

$$\sigma N_k(u) \subset [N_k^l, N_k^r], \quad u \in S_0^\infty \tag{62}$$

for each $k = 1, \dots, M$. Assume that we are also have some bounds

$$\epsilon f_k(t) \subset [f_k^l, f_k^r], \quad t \in \mathbb{R}. \tag{63}$$

Given these enclosures, inequalities (61) can be checked easily, in our case on a computer using interval arithmetics. Note that there is no guarantee that for given $\epsilon \neq 0$ and given bounds S_0^∞ the inequalities (61) will be satisfied. However, for $|\epsilon|$ small enough "good" bounds should exist. Since we cannot expect to choose the correct values for $\{u_k^{l,r}\}$, C and s at first try, we will use an algorithm from [21] (Section 3.3) for refining an initial guess for the bounds.

Our goal is both to increase s and correct our guesses for the bounds on coordinates where the isolation inequalities do not hold. We iteratively adjust the pairs (u_k^l, u_k^r), and later C and s, so at each step our new guess is at worst case an equality in the isolation conditions. This way the bounds are tight on each coordinate, so the nonlinear terms do not contribute much error. Note that the procedure is heuristic and we do not claim that the algorithm will produce correct bounds – this we verify a posteriori in interval arithmetics.

1. First we adjust C and s. Recall that, by (42) for $k > M$ we want to choose C and s such that

$$\frac{C}{k^s} > \frac{\sigma D}{2k^{s+1}(\beta - k^{-2})} \,. \tag{64}$$

We want to increase s - therefore, we set the new parameters by

$$s := s + 1 \,,$$
$$C := \frac{\sigma D}{2(\beta - (M+1)^{-2})} \,. \tag{65}$$

2. Trying to comply with (61) we set the new u_k^l's and u_k^r's inductively for $k = 1, \ldots, M$ by

$$u_k^r := -\frac{N_k^l + k^{-2} f_k^l}{2(\beta k^2 - 1)} \,,$$
$$u_k^l := -\frac{N_k^r + k^{-2} f_k^r}{2(\beta k^2 - 1)} \,. \tag{66}$$

After each run we check inequalities (61) and (35) to see whether we obtained isolation. We may however require additional iterates to improve s.

4.5. Proof of Theorem 1

For each of the parameter values and forcing terms we take a guess on the initial bounds with higher modes decay of order $\tilde{C}/k^4$. Once a guess for some given range of ϵ is found it is easy to adapt it to another ϵ by rescaling proportionally to the upper bound on $|\epsilon|$. After two iterates of procedure given in Subsection 4.4 we obtain the values of $\{u_k^{l,r}\}$, C and $s = 6$ such that inequalities (61) and (35) hold. We present the approximate values (the first 5 significant digits of the actual values) in Table 3. We remark that in all of the cases it was enough to take $M = 6$. From Corollary (8)

Tab 3.: Parameters M and s and approximate values of C, u_k^l and u_k^r, $k = 1, \ldots, M$ used in the proof of Theorem 1

$f \in$	$\mathcal{F}_\tau^A$			$\mathcal{F}_\tau^B$		
β	1.5	1.75	2.5	1.5	1.75	2.5
ϵ	$[-0.05, 0.05]$	$[-0.1, 0.1]$	$[-0.3, 0.3]$	$[-0.05, 0.05]$	$[-0.1, 0.1]$	$[-0.3, 0.3]$
M	6	6	6	6	6	6
$u_1^r = -u_1^l$	0.05743	0.082489	0.16777	0.059242	0.085611	0.17515
$u_2^r = -u_2^l$	0.004018	0.0069984	0.020237	0.0055667	0.0097091	0.026475
$u_3^r = -u_3^l$	0.00022427	0.0004798	0.0019934	0.00054628	0.0010739	0.0035043
$u_4^r = -u_4^l$	1.1242×10^{-5}	2.9597×10^{-5}	0.00017727	9.3307×10^{-5}	0.000179	0.00055121
$u_5^r = -u_5^l$	5.7862×10^{-7}	1.9415×10^{-6}	1.8646×10^{-5}	2.9174×10^{-6}	7.6705×10^{-6}	4.3434×10^{-5}
$u_6^r = -u_6^l$	5.5904×10^{-7}	1.9328×10^{-6}	2.1631×10^{-5}	7.7255×10^{-7}	2.596×10^{-6}	2.6004×10^{-5}
C	4.6941	13.039	100.64	4.8878	13.613	102.99
s	6	6	6	6	6	6

we conclude that for each Galerkin projection (30) for $n > M$ there exists a periodic solution $(u^{+,n}(t), u^{-,n}(t))$ such that

$$u^{\pm,n}(t) \in \prod_{k=1}^{M} [u_k^l, u_k^r] \oplus \prod_{k=M+1}^{n} [-C/k^s, C/k^s], \quad \forall t \in \mathbb{R}. \tag{67}$$

The above computations were done on a computer in interval arithmetics, as it would be tedious to do them by hand. The source files are available online [7]. The program uses interval arithmetics implementation from the CAPD package [1].

By the change of variables (26) we return now to the original coordinates u_k, v_k, i.e. the Fourier coefficients of u and u_t and obtain a sequence of periodic solutions $(u^n(t), v^n(t))$ of the Galerkin projections of the system (25). From equation (67) and the form of our change of variables it follows, that there exists a $\hat{C} > 0$ (an exact value of which is not important to us), such that

$$u_k^n(t) \leq \frac{\hat{C}}{k^6}, \qquad v_k^n(t) \leq \frac{\hat{C}}{k^4}. \tag{68}$$

for all $n, k : k \leq n$ and $t \in \mathbb{R}$. A standard argument (cf. Theorem 10 in [20]) proves that the set

$$\prod_{k=1}^{\infty} [-\hat{C}/k^6, \hat{C}/k^6] \times [-\hat{C}/k^4, \hat{C}/k^4] \tag{69}$$

satisfies conditions **C2**, **C3** and forms self-consistent bounds for (25). Note that at this moment we need the polynomial coefficient decay rate to be of order at least 2 for v_k^n's and 6 for u_k^n's (we have 4 and 6, respectively). Let $u^n(t, x) = \sum_{k \in \mathbb{N}} u_k(t) e^{ikx}$, $v^n(t, x) = \sum_{k \in \mathbb{N}} v_k(t) e^{ikx}$. From Theorem 5 it follows that the sequence $\{(u^n(t, x), v^n(t, x))\}_n$ has a subsequence $\{(u^{n_l}(t, x), v^{n_l}(t, x))\}_l$ converging uniformly on compact time intervals to a solution $(u^*(t, x), v^*(t, x))$ of

$$
\begin{aligned}
u_t &= v, \\
v_t &= u_{xx} + \beta u_{xxxx} + \sigma(u^2)_{xx} + \epsilon f(t, x),
\end{aligned}
\tag{70}
$$

i.e. the Boussinesq equation (1) rewritten as a first order system. We have

$$u^*(t, x) = \lim_{l \to \infty} u^{n_l}(t, x) = \lim_{l \to \infty} u^{n_l}(t + \tau, x) = u^*(t + \tau, x) \tag{71}$$

for all $x, t \in \mathbb{R}$, hence the solution is periodic.

The C^0 and L^2 bounds on u^* and v^* are computed from equations (26) and (67). From the coefficient decay (68) and elementary facts about the Fourier series (see Section 6 in [18]) it follows that $u^*(t, x)$ is of class C^4 and $v^*(t, x)$ is of class C^2 as functions of x.

5. References

[1] CAPD: Computer Assisted Proofs in Dynamics, a Package for Rigorous Numerics. `http://capd.ii.uj.edu.pl`.

[2] G. Arioli and H. Koch. Computer-assisted methods for the study of stationary solutions in dissipative systems, applied to the Kuramoto-Sivashinski equation. *Arch. Ration. Mech. Anal.*, 197(3):1033–1051, 2010.

[3] G. Arioli and H. Koch. Integration of dissipative partial differential equations: a case study. *SIAM J. Appl. Dyn. Syst.*, 9(3):1119–1133, 2010.

[4] J. Boussinesq. Thorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pure Appl.*, 17(2):55–108, 1872.

[5] J. Cyranka. Existence of Globally Attracting Fixed Points of Viscous Burgers Equation with Constant Forcing. A Computer Assisted Proof. *Topol. Methods Nonlinear Anal.*, to appear.

[6] J. Cyranka and P. Zgliczyński. Existence of Globally Attracting Solutions for One-Dimensional Viscous Burgers Equation with Nonautonomous Forcing—A Computer Assisted Proof. *SIAM J. Appl. Dyn. Syst.*, 14(2):787–821, 2015.

[7] A. Czechowski. personal home page. `http://www.ii.uj.edu.pl/~czechows`.

[8] S. Day, J.-P. Lessard, and K. Mischaikow. Validated continuation for equilibria of PDEs. *SIAM J. Numer. Anal.*, 45(4):1398–1424, 2007.

[9] M. Gameiro and J.-P. Lessard. Analytic estimates and rigorous continuation for equilibria of higher-dimensional PDEs. *J. Differential Equations*, 249(9):2237–2268, 2010.

[10] R. Hirota. Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices. *J. Mathematical Phys.*, 14:810–814, 1973.

[11] V. S. Manoranjan, A. R. Mitchell, and J. L. Morris. Numerical solutions of the good Boussinesq equation. *SIAM J. Sci. Statist. Comput.*, 5(4):946–957, 1984.

[12] R. Srzednicki. Periodic and constant solutions via topological principle of Ważewski. *Univ. Iagel. Acta Math.*, (26):183–190, 1987.

[13] R. Srzednicki. Periodic and bounded solutions in blocks for time-periodic nonautonomous ordinary differential equations. *Nonlinear Anal.*, 22(6):707–737, 1994.

[14] R. Srzednicki and K. Wójcik. A geometric method for detecting chaotic dynamics. *J. Differential Equations*, 135(1):66–82, 1997.

[15] V. E. Zakharov. On stochastization of one-dimensional chains of nonlinear oscillators. *Sov. Phys.-JETP*, 38:108–110, 1974.

[16] P. Zgliczyński. Steady state bifurcations for the Kuramoto-Sivashinsky equation – a computer assisted proof. *AIMS J. of Comp. Dyn.*, to appear.

[17] P. Zgliczynski. Attracting fixed points for the Kuramoto-Sivashinsky equation: a computer assisted proof. *SIAM J. Appl. Dyn. Syst.*, 1(2):215–235, 2002.

[18] P. Zgliczyński. Trapping regions and an ODE-type proof of the existence and uniqueness theorem for Navier-Stokes equations with periodic boundary conditions on the plane. *Univ. Iagel. Acta Math.*, (41):89–113, 2003.

[19] P. Zgliczyński. Rigorous numerics for dissipative partial differential equations. II. Periodic orbit for the Kuramoto-Sivashinsky PDE—a computer-assisted proof. *Found. Comput. Math.*, 4(2):157–185, 2004.

[20] P. Zgliczyński. Rigorous numerics for dissipative PDEs III. An effective algorithm for rigorous integration of dissipative PDEs. *Topol. Methods Nonlinear Anal.*, 36(2):197–262, 2010.

[21] P. Zgliczyński and K. Mischaikow. Rigorous numerics for partial differential equations: the Kuramoto-Sivashinsky equation. *Found. Comput. Math.*, 1(3):255–288, 2001.