Proper holomorphic mappings, Bell's formula and the Lu Qi-Keng problem on the tetrablock

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Abstract

We consider proper holomorphic maps $\pi : D \to G$ where D and G are domains in $\mathbb{C}^n$. Let $\alpha \in \mathcal{C}(G, \mathbb{R}_{>0})$. We show that every π induces some subspace H of $\mathbb{A}^2_{\alpha\circ\pi}(D)$ such that $\mathbb{A}^2_\alpha(G)$ is isometrically isomorphic with H via some unitary operator Γ. Using this isomorphism we construct the orthogonal projection onto H and we derive Bell's transformation formula for the weighted Bergman kernel function under proper holomorphic mappings. As a consequence of the formula we get that the tetrablock is not a Lu Qi-Keng domain.

1 Introduction

Misra, Roy, and Zhang [12] recently studied the pullback of the Bergman space under a proper holomorphic mapping in the context of the symmetrized polydisc. Here we generalize the construction to arbitrary domains. In particular, we obtain a new proof of Bell's transformation formula for the Bergman kernel function under proper holomorphic mappings. As an application, we demonstrate that the Bergman kernel function of the tetrablock has zeros.

2 Construction of the operator Γ

Let D and G be domains in $\mathbb{C}^n$. Let $\pi : D \to G$ be a proper holomorphic map with multiplicity m, and fix any $\alpha \in \mathcal{C}(G, \mathbb{R}_{>0})$. By $\mathbb{A}^2_\alpha(G)$ we understand the space of all square integrable holomorphic functions on G with respect to the weight function α, that is

$$\mathbb{A}^2_\alpha(G) = \{f \in \mathcal{O}(G) : \int_G |f|^2 \alpha \, dV < \infty\}.$$

The space $\mathbb{A}^2_\alpha(G)$ with the scalar product

$$\langle f, g\rangle_{\mathbb{A}^2_\alpha(G)} = \int_G f(z)\overline{g(z)}\alpha(z)\, dV(z), f, g \in \mathbb{A}^2_\alpha(G),$$

is a complex Hilbert space, the Hilbert space of all square integrable holomorphic functions on G with respect to the weight function α. Let $J\pi$ denote the complex Jacobian of π. We show that there is some closed subspace H of $\mathbb{A}^2_{\alpha\circ\pi}(D)$ (closely related with π) which is unitary isomorphic to $\mathbb{A}^2_\alpha(G)$. We also derive an explictit formula for the orthogonal projection onto H.

We proceed to formulate the most important component of this paper.

Let

$$\Gamma : \mathbb{A}^2_\alpha(G) \to \mathbb{A}^2_{\alpha\circ\pi}(D),$$

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be defined as follows

$$\Gamma f = \frac{1}{\sqrt{m}}(f \circ \pi) J\pi, \quad f \in \mathbb{A}^2_\alpha(G).$$

Γ's adjoint operator is of great importance to us, so we explain how it works. In fact, Γ^* equals its inverse Γ^{-1} (if Γ is understood as an operator from $\mathbb{A}^2_\alpha(G)$ onto $\Gamma\mathbb{A}^2_\alpha(G)$). To describe Γ^* take any $g \in \Gamma\mathbb{A}^2_\alpha(G)$. Then $\frac{g}{J\pi}$ is a well-defined function on a dense, open subset of D (the set of regular points of π). Moreover, notice that $\frac{g}{J\pi}$ is invariant under π, that is $\frac{g}{J\pi}(z) = \frac{g}{J\pi}(w)$ for any $z, w \in D$ such that $\pi(z) = \pi(w)$, $J\pi(w), J\pi(z) \neq 0$. Therefore, equality $\widetilde{(\frac{g}{J\pi})}(\pi(z)) = \frac{g}{J\pi}(z)$ defines well a holomorphic function on G except for the (analytic) set of critical values of π. However, the Riemann removable singularity Theorem for square integrable holomorphic functions (see e.g. [8], Theorem 4.2.9) ensures that $\widetilde{(\frac{g}{J\pi})}$ has a holomorphic extension on D (denoted by the same symbol). After this consideration the adjoint operator to Γ might be described by equality

$$\Gamma^* g = \sqrt{m}\widetilde{(\frac{g}{J\pi})}, \quad g \in \Gamma\mathbb{A}^2_\alpha(G).$$

The preceding assumptions remain in force below, unless otherwise stated.

3 Main results

Theorem 1. *The set $\Gamma\mathbb{A}^2_\alpha(G)$ is a closed subspace of $\mathbb{A}^2_{\alpha\circ\pi}(D)$, that is isometrically isomorphic with $\mathbb{A}^2_\alpha(G)$ via Γ. The orthogonal projection P onto $\Gamma\mathbb{A}^2_\alpha(G)$ is given by a formula*

$$Pg = \frac{1}{m}\sum_{k=1}^{m}(g \circ \pi^k \circ \pi) J(\pi^k \circ \pi), \quad g \in \mathbb{A}^2_{\alpha\circ\pi}(D),$$

where $\{\pi^j\}_{j=1}^m$ are the local inverses to π.

Note that it will follow from the proof that the formula on the right side actually defines a function from $\Gamma\mathbb{A}^2_\alpha(G) \subset \mathbb{A}^2_{\alpha\circ\phi}(D)$.

Remark 1. *In [8] the Riemann removable singularity theorem is proved for the case $\alpha \equiv 1$, but this proof might be repeated without any trouble in case $\alpha \in \mathcal{C}(G, \mathbb{R}_{>0})$. In fact, local boundedness of α is sufficient.*

Motivated by [11], in this paper we investigate the relations between weighted Bergman Spaces: $\mathbb{A}^2_\alpha(G)$ and $\mathbb{A}^2_{\alpha\circ\pi}(D)$. Recall that the *Bergman kernel function with weight α* of a domain G (denoted K_α) is the reproducing kernel of the space $\mathbb{A}^2_\alpha(G)$. Using the Cauchy integral formula it follows that for every $z \in G$ the evaluation functional $ev_z : \mathbb{A}^2_\alpha(G) \ni f \to f(z) \in \mathbb{C}$ is continuous. Thus from the Riesz representation Theorem there is the unique function $K^\alpha_{G,z} \in \mathbb{A}^2_\alpha(G)$ (called the kernel function) such that $ev_z(f) = \langle f, K^\alpha_{G,z}\rangle_{\mathbb{A}^2_\alpha(G)}$. Then the *Bergman kernel function with weight α* might be written as follows

$$K^\alpha_G(z, w) = \langle K^\alpha_{G,w}, K^\alpha_{G,z}\rangle_{\mathbb{A}^2_\alpha(G)}, \quad z, w \in G.$$

For $\alpha \equiv 1$ we simply write $K^\alpha_G = K_G$, and call it the *Bergman kernel function*. The definition and basic properties of the Bergman kernel function might be found in [7].

As a corollary of Theorem 1 we get Bell's Theorem. Originally Bell formulated transformation rule for the Bergman kernel function with weight $\alpha \equiv 1$. Here we shall prove that the same formula holds in more general setting, which seems not to have been noticed in the literature.

Corollary 1 (see [2])**.** *Let D and G be domains in $\mathbb{C}^n$ and let $\pi : D \to G$ be a proper holomorphic map with multiplicity m. Denote by $\pi^1, \ldots, \pi^m$ the local inverses of π. Then*

$$\overline{J\pi(w)} K^\alpha_G(\pi(z), \pi(w)) = \sum_{k=1}^{m} K^{\alpha\circ\pi}_D(\pi^k \circ \pi(z), w) J\pi^k(\pi(z)), \quad \text{for any } z \notin \pi^{-1}(\pi(N(J\pi))),$$

where $N(J\pi) = \{J\pi = 0\}$.

Remark 2. *Let $\alpha \equiv 1$. The proof of Theorem 1 shows that Γ is the restriction of an operator Γ_e from $L^2(G)$ to $L^2(D)$, given by the same formula, and all statements contained in Theorem 1 hold for Γ_e. Moreover, this together with the transformation formula for the Bergman projection operator, given in [2], allow us to write $P_D\Gamma_e = \Gamma_e P_G$, where P_G and P_D denote the Bergman projections of the domains D and G.*

Proof of Theorem 1. The idea of the formula of Γ was inspired by the rule

$$m\int_G f\alpha\, dV = \int_D (f\circ\pi)|J\pi|^2(\alpha\circ\pi)\, dV \quad \text{for any } f\in \mathrm{L}^1_\alpha(G) = \{g : G\to\mathbb{C} : \int_G |f|\alpha\, dV < \infty\},$$

which makes the Γ an isometry. The above rule ensures that the range of Γ is a closed Hilbert subspace of $\mathbb{A}^2_{\alpha\circ\pi}(D)$. Therefore, Γ is a unitary operator from $\mathbb{A}^2_\alpha(G)$ onto $\Gamma\mathbb{A}^2_\alpha(G)$.

Thus, there is the orthogonal projection P from $\mathbb{A}^2_{\alpha\circ\pi}(D)$ onto $\Gamma\mathbb{A}^2_\alpha(G)$. We prove that P is given by the formula

$$Pg = \frac{1}{m}\sum_{k=1}^m (g\circ\pi^k\circ\pi)J(\pi^k\circ\pi), \quad g\in\mathbb{A}^2_{\alpha\circ\pi}(D).$$

Let us denote the right side by Qg. First of all, we need to show that Q is well defined. Using the properness of π one can easily compute

$$\|Qg\|^2_{\mathbb{A}^2_{\alpha\circ\pi}(D)} = \frac{1}{m^2}\int_D \Big|\sum_{k=1}^m (g\circ\pi^k\circ\pi)J(\pi^k\circ\pi)\Big|^2(\alpha\circ\pi)dV \leq$$
$$\frac{1}{m}\int_D \sum_{k=1}^m |(g\circ\pi^k\circ\pi)J(\pi^k\circ\pi)|^2(\alpha\circ\pi)dV = \|g\|^2_{\mathbb{A}^2_{\alpha\circ\pi}(D)},$$

for $g\in\mathbb{A}^2_{\alpha\circ\pi}(D)$. It remains to verify whether Qg is holomorphic. For that fix some $g\in\mathbb{A}^2_{\alpha\circ\pi}(D)$. Notice that the map $\frac{Qg}{J\pi}$ is a well defined holomorphic function on a set $D\setminus\pi^{-1}(\pi(N(J\pi)))$, constant on the fibres of π. So, it induces some map $\widetilde{(\frac{Qg}{J\pi})}$ which is holomorphic on $G\setminus\pi(N(J\pi))$. The Riemann removable singularity Theorem (see Remark 1) finishes the correctness of the definition of Q provided we know that $\widetilde{(\frac{Qg}{J\pi})}$ is square integrable with weight α on G. But for that it is enough to show that $Qg\in\mathbb{A}^2_{\alpha\circ\pi}(D)$ what we have just proved. Actually, we have established something more. Namely, that for any $g\in\mathbb{A}^2_{\alpha\circ\pi}(D)$ the equation $Qg = \Gamma f$ has solution f in $\mathbb{A}^2_\alpha(G)$. (The application of the Riemann removable singularity theorem on the domain D would give us only that $Qg\in\mathbb{A}^2_{\alpha\circ\pi}(D)$.)

Secondly, notice that $Q^2 = Q$. Indeed,

$$\begin{aligned}
Q^2 g = \frac{1}{m}\sum_{l=1}^m (Qg\circ\pi^l\circ\pi)J(\pi^l\circ\pi) &= \frac{1}{m^2}\sum_{l=1}^m\sum_{k=1}^m (g\circ\pi^l\circ\pi\circ\pi^k\circ\pi)[J(\pi^l\circ\pi)\circ\pi^k\circ\pi]J(\pi^k\circ\pi)\\
&= \frac{1}{m^2}\sum_{k=1}^m\sum_{l=1}^m (g\circ\pi^l\circ\pi\circ\pi^k\circ\pi)[J(\pi^l\circ\pi)\circ\pi^k\circ\pi]J(\pi^k\circ\pi)\\
&= \frac{1}{m^2}\sum_{k=1}^m\sum_{l=1}^m (g\circ\pi^l\circ\pi)J(\pi^l\circ\pi\circ\pi^k\circ\pi) = \frac{1}{m^2}\sum_{k=1}^m\sum_{l=1}^m (g\circ\pi^l\circ\pi)J(\pi^l\circ\pi) = Qg,
\end{aligned}$$

for $g\in\mathbb{A}^2_{\alpha\circ\pi}(D)$.

Up to this point, we only know that Q is the projection. Next, we proceed to show the equality $\operatorname{ran}\Gamma = \operatorname{ran} Q$. Similarly as above we get $Q\circ\Gamma = \Gamma$, which gives "$\subset$". It remains to demonstrate the opposite inclusion. So, the question is whether Q takes values in $\Gamma\mathbb{A}^2_\alpha(G)$. Since $Q^2 = Q$, it is enough to show that for any $g\in\mathbb{A}^2_{\alpha\circ\pi}(D)$ the equation $Qg = \Gamma f$ has solution f in $\mathbb{A}^2_\alpha(G)$, and it holds as we proved it before. Finally, since $Q^2 = Q$, $Q|_{\operatorname{ran} Q} = \mathrm{id}|_{\operatorname{ran} Q}$ and Q is bounded, Q is the orthogonal projection onto $\Gamma\mathbb{A}^2_{\alpha\circ\pi}(D)$. $\square$

Proof of Corollary 1. We keep the notation from the previous proof. Keeping in mind the discussion from the second section and the proof of Theorem 1, observe that the reproducing property of the weighted

Bergman kernel function implies that for any $f \in \mathbb{A}^2_\alpha(G)$ and $w \in D$ the following equalities hold

$$\langle \Gamma f, PK_D^{\alpha\circ\pi}(\cdot, w)\rangle_{\mathbb{A}^2_{\alpha\circ\pi}(D)} = \langle \Gamma f, K_D^{\alpha\circ\pi}(\cdot, w)\rangle_{\mathbb{A}^2_{\alpha\circ\pi}(D)} = \Gamma f(w) = \frac{1}{\sqrt{m}} f(\pi(w)) J\pi(w)$$
$$= \langle f, K_G^\alpha(\cdot, \pi(w))\rangle_{\mathbb{A}^2_\alpha(G)} \frac{J\pi(w)}{\sqrt{m}} = \langle \Gamma f, \Gamma K_G^\alpha(\cdot, \pi(w))\rangle_{\mathbb{A}^2_{\alpha\circ\pi}(D)} \frac{J\pi(w)}{\sqrt{m}}.$$

Consequently, from the Riesz representation Theorem (uniqueness), applied to the space $\Gamma\mathbb{A}^2_\alpha(G)$, and the unitarity of Γ we get

$$\overline{J\pi(w)} K_G^\alpha(\pi(\cdot), \pi(w)) = \sqrt{m}\, (\Gamma^* \circ P) K_D^{\alpha\circ\pi}(\cdot, w)(\pi(\cdot)).$$

The last equality holds on $D \setminus \pi^{-1}(\pi(N(J\pi)))$ for arbitrary $w \in D$. But if we take $w \notin \pi^{-1}(\pi(N(J\pi)))$, then on the same set we have

$$K_G^\alpha(\pi(\cdot), \pi(w)) = (\Gamma^* \circ P) \frac{\sqrt{m}}{\overline{J\pi(w)}} K_D^{\alpha\circ\pi}(\cdot, w).$$

Unwinding the definitions of Γ^* and P produces the desired statement. $\square$

4 The tetrablock is not a Lu Qi-Keng domain

Recall the definition of the object which name appears in the title of this section.

Let
$$\varphi : \mathcal{R}_{II} \to \mathbb{C}^3, \ \varphi(z_{11}, z_{22}, z) := (z_{11}, z_{22}, z_{11}z_{22} - z^2),$$
where $\mathcal{R}_{II}$ denotes the classical Cartan domain of the second type (in $\mathbb{C}^3$), that is

$$\mathcal{R}_{II} = \{\widetilde{z} \in \mathcal{M}_{2\times 2}(\mathbb{C}) : \widetilde{z} = \widetilde{z}^t, \ \|\widetilde{z}\| < 1\},$$

where $\|\cdot\|$ is the operator norm and $\mathcal{M}_{2\times 2}(\mathbb{C})$ denotes the space of 2×2 complex matrices (we identify a point $(z_{11}, z_{22}, z) \in \mathbb{C}^3$ with a 2×2 symmetric matrix $\begin{pmatrix} z_{11} & z \\ z & z_{22} \end{pmatrix}$). Then φ is a proper holomorphic map and $\varphi(\mathcal{R}_{II}) = \mathbb{E}$ is a domain (see Remark 4 below), called the *tetrablock*.

The tetrablock was first studied in [1]. Afterwards it was studied by many authors. In particular, it was shown that the tetrablock is a $\mathbb{C}$-convex domain (see [15]). The importance of the tetrablock for the geometric function theory follows from the fact that it is the second example (the first one was was the symmetrized bidisc) which is hyperconvex and not biholomorphically equivalent to a convex domain but despite it the Lempert Theorem (see [9] and [10]) holds for it (see [5]). It is also natural to find the Bergman kernel function for the tetrablock (using the formula for the Bergman kernel function of the Cartan domain and Bell's transformation formula). To our surprise it turned out that the tetrablock is not a Lu Qi Keng domain. Moreover, it vanishes at very simple points. Recall that a domain D is *a Lu Qi-Keng domain* if its Bergman kernel function with weight $\alpha \equiv 1$ does not have zeros and is *not a Lu Qi-Keng domain* if it has.

As to the history of the Lu Qi Keng problem we refer the interested Reader to [3]. There are many results on both : domains being a Lu Qi-Keng and being not a Lu Qi-Keng (see e.g. in [4], [14]) .

Recall that ([6] p. 84)

$$K_{\mathcal{R}_{II}}(t, s) = \frac{1}{\mathrm{Vol}(\mathcal{R}_{II})} \Big(\det(I - t\overline{s})\Big)^{-3}, \quad \text{for } t,\ s \in \mathcal{R}_{II}.$$

Since every point in $\mathcal{R}_{II}$ can be carried by some automorphism of $\mathcal{R}_{II}$ into the origin (see [6] p. 84), we get $K_{\mathcal{R}_{II}} \neq 0$. Thus, $\mathcal{R}_{II}$ is a Lu Qi-Keng domain. Therefore, we have a proper holomorphic mapping $\varphi : \mathcal{R}_{II} \to \mathbb{E}$ of multiplicity 2 such that $\mathcal{R}_{II}$ is a Lu Qi-Keng domain whereas $\mathbb{E}$ is not a Lu Qi-Keng domain. Recall that another example of that type is $\{|z|+|w| < 1\} \ni (z, w) \to (z^2, w) \in \{|z|^{\frac{1}{2}} + |w| < 1\}$ (see [4]). In our situation there is equality of holomorphically invariant distances in both domains and both domains are $\mathbb{C}$-convex (see [5], [15]) whereas in the example from [4] it is not the case.

Below we show two results which are consequences of Bell's transformation formula.

Corollary 2. *For any* $\widetilde{z}=(z_{11},z_{22},z)$, $\widetilde{w}=(w_{11},w_{22},w)\in\mathcal{R}_{II}$

$$J\varphi(\widetilde{z})\,K_{\mathbb{E}}(\varphi(\widetilde{z}),\varphi(\widetilde{w}))\,\overline{J\varphi(\widetilde{w})}=K_{\mathcal{R}_{II}}\Big((z_{11},z_{22},z),\widetilde{w}\Big)-K_{\mathcal{R}_{II}}\Big((z_{11},z_{22},-z),\widetilde{w}\Big).$$

A consequence of the last formula is the following:

Corollary 3. $\mathbb{E}$ *is not a Lu Qi-Keng domain.*

We set about achieving above Corollaries.

Proof of Corollary 2. Below we present how operators: Γ and P work in some very special case, that is when $\pi=\varphi$, $\alpha=1$, $D=\mathcal{R}_{II}$, $G=\mathbb{E}$. It is not necessary to do that to write down a formula for $K_{\mathbb{E}}$, but it is so simple in that case, that we think it is worth stating. (We keep the notation from the second section) The range of the operator Γ is contained in the set of those maps whose coefficients at $z_{11}^k z_{22}^l z^{2n}$ in the Taylor expansion at the origin vanish for all k, l, n natural numbers. We showed that every function in $\Gamma\mathbb{A}^2(\mathbb{E})$ is of the form $J\pi\cdot h$ for some function h depending on z_{11}, z_{22}, z^2, but not necessarily conversly. The projection

$$P:\mathbb{A}^2(\mathcal{R}_{II})\to\Gamma\mathbb{A}^2(\mathbb{E}),$$

acts as follows

$$P(f)(z_{11},z_{22},z)=\frac{1}{2}\Big(f(z_{11},z_{22},z)-f(z_{11},z_{22},-z)\Big),\quad f\in\mathbb{A}^2(\mathcal{R}_{II}),\ (z_{11},z_{22},z)\in\mathcal{R}_{II},$$

and the adjoint

$$\Gamma^*g=\sqrt{2}\widetilde{\Big(\frac{g}{\mathrm{J}\varphi}\Big)},\quad g\in\Gamma\mathbb{A}^2(\mathbb{E}).$$

From the proof of Collorary 1, we might write

$$K_{\mathbb{E}}(\varphi(\cdot),\varphi(w_{11},w_{22,}w))=(\Gamma^*\circ P)\frac{\sqrt{2}}{\overline{J\varphi(w_{11},w_{22},w)}}K_{\mathcal{R}_{II}}(\cdot,(w_{11},w_{22},w)),\quad\text{for }(w_{11},w_{22},w)\notin N(J\varphi),$$

and finally

$$K_{\mathbb{E}}(\varphi(z_{11},z_{22},z),\varphi(w_{11},w_{22},w))=$$
$$\frac{K_{\mathcal{R}_{II}}\Big((z_{11},z_{22},z),(w_{11},w_{22},w)\Big)-K_{\mathcal{R}_{II}}\Big((z_{11},z_{22},-z),(w_{11},w_{22},w)\Big)}{J\varphi(z_{11},z_{22},z)\overline{J\varphi(w_{11},w_{22},w)}},$$

for (z_{11},z_{22},z), $(w_{11},w_{22},w)\notin N(J\varphi)$. □

Proof of Corollary 3. We examine the formula for the Bergman kernel function for $\mathbb{E}$ for pair $\varphi(0,0,1)$, $\varphi(0,0,z)$ (note that the formula for the Bergman kernel function for $\mathcal{R}_{II}$ extends analytically to $\overline{\mathcal{R}_{II}}\times\mathcal{R}_{II}$). Calculation shows that $K_{\mathbb{E}}(\varphi(0,0,1),\varphi(0,0,z))=\frac{\pi^3}{6}(3+10\overline{z}^2+3\overline{z}^4)(1-\overline{z}^2)^{-6}$, $z\in\mathbb{D}$, and the last expression vanishes for $z_0^2=-\frac{1}{3}$. Now the equality

$$K_{\mathbb{E}}(\varphi(0,0,1),\varphi(0,0,z_0))=K_{\mathbb{E}}(\varphi(0,0,r),\varphi(0,0,\frac{1}{r}z_0)),$$

which holds for $0<r<1$ such that $\frac{z_0}{r}\in\mathbb{D}$, finishes the proof. □

5 Remark on proper holomorphic mappins

One may show directly the fact that φ described in the last section is a proper holomorphic mapping and $\mathbb{E}$ is a domain. However, it seems reasonable to formulate a result which will be the generalization of that fact and that will help us to avoid the ad hoc proof of properness and openness of a wide class of mappings. That is the reason why we present below some auxiliary result whose idea of the proof basically comes from the proof of Proposition 2.1 in [13]. We formulate and show it in a more general setting so that it could be applied among others to the above mentioned case of the tetrablock and the symmetrized polydisc.

Proposition 1. *Let D be a domain in $\mathbb{C}^n$. Let $\pi : D \to \mathbb{C}^n$ be a holomorphic map. Assume there exists a finite group of homeomorphic transformations $\mathcal{U}$ of D such that π is precisely $\mathcal{U}$-invariant, that is for $z, w \in D$ we have that $\pi(z) = \pi(w)$ if and only if $Uz = w$ for some $U \in \mathcal{U}$. Then $\pi(D)$ is a domain and $\pi : D \to \pi(D)$ is a proper mapping.*

Proof of Proposition 1. Let $\{K_k\}_{k\in\mathbb{N}}$ be an increasing sequence of relatively compact domains of D exhausting D. Consider a new sequence $\{D_k := \bigcup_{U\in\mathcal{U}} U(K_k)\}_k$. Since $\mathcal{U}$ is finite, the set D_k is a relatively compact subset of D for every k. Moreover, there is some N such that for $k > N$ the set D_k is a domain. Certainly, this new sequence $\{D_k\}_k$ is exhausting D. Fix $k > N$. Then $\mathcal{U}_k = \{U|_{D_k} : U \in \mathcal{U}\}$ is a finite group of automorphisms of D_k and $\pi|_{D_k} : D_k \to \mathbb{C}^n$ is precisely $\mathcal{U}_k$-invariant. These two facts together with the properness of $U|_{D_k}$ as a selfmap of D_k for every $U \in \mathcal{U}$, imply that the intersection of the sets $\pi(D_k)$ and $\pi(\partial D_k)$ is empty. Let Ω_k be the component of $\mathbb{C}^n \setminus \pi(\partial D_k)$ that contains $\pi(D_k)$. Consequently, we get that $\pi(\partial D_k) \subset \partial\Omega_k$. This implies that $\pi|_{D_k} : D_k \to \Omega_k$ is a proper map (here we used the fact that π is a holomorphic map on D_k which extends continuously to $\overline{D_k}$).

Therefore, $\Omega_k = \pi(D_k)$. Let $\Omega = \cup_k \Omega_k$. Evidently, $\Omega = \pi(D)$ is a domain in $\mathbb{C}^n$. The properness of π might be checked as follows. If $K \subset \Omega$ is compact, then $K \subset \Omega_k$ for some k. Hence $\pi^{-1}(K)$ is a compact subset of D_k, and thus a compact subset of Ω. □

Remark 3. *In Proposition 1 we only assumed that every $U \in \mathcal{U}$ is a homeomorphism but the equality $\pi \circ U = \pi$ easily implies that $\mathcal{U}$ actually is necessarily contained in the group of holomorphic automorphisms of D.*

Remark 4. *Map φ (defined above) is $\mathcal{U}_{\mathbb{E}} = \{\mathrm{Id}, \mathrm{diag}(1, 1, -1)\}$-invariant. What needs be to verified is only whether $\mathcal{U}_{\mathbb{E}}$ describes a subgroup of the group of automorphisms of $\mathcal{R}_{II}$. It can be derived by showing that the norm of matrix $\begin{pmatrix} z_{11} & z \\ z & z_{22} \end{pmatrix}$ (vieved as an operator on $\mathbb{C}^2$) equals the norm of the related matrix $\begin{pmatrix} z_{11} & -z \\ -z & z_{22} \end{pmatrix}$. But the norm of the matrix $\begin{pmatrix} z_{11} & -z \\ -z & z_{22} \end{pmatrix}$ equals*

$$\sup_{a,b,c,d\in\mathbb{C},|a|^2+|b|^2=1|c|^2+|d|^2=1} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} z_{11} & -z \\ -z & z_{22} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix},$$

and clearly

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} z_{11} & -z \\ -z & z_{22} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a & -b \end{pmatrix} \begin{pmatrix} z_{11} & z \\ z & z_{22} \end{pmatrix} \begin{pmatrix} c \\ -d \end{pmatrix},$$

so the claim follows.

Remark 5. *Let us consider a map $\pi = (\pi_1, \ldots, \pi_n)$ on $\mathbb{D}^n$ where π_j is the j-th elementary symmetric polynomial. In that case the finite group of unitary transformations under which π is precisely invariant is the group of permutations $\mathcal{S}_n$. Proposition 1 gives the proof of the fact that $\pi|_{\mathbb{D}^n}$ is a proper holomorphic mapping onto the image i. e. the symmetrized polydisc and the symmetrized polydisc is open.*

Remark 6. *Fix any $k > 2$ and consider: a function $\varphi_k : \mathcal{R}_{II} \to \mathbb{C}^3$, $\varphi_k(z_{11}, z_{22}, z) = (z_{11}, z_{22}, z_{11}z_{22} - z^k)$ and a set $\varphi_k(\mathcal{R}_{II})$. Notice that φ is not proper onto its image. If it were, then a map $\varphi_\zeta : \mathcal{R}_{II} \to \mathcal{M}_{2\times 2}(\mathbb{C})$, $(z_1, z_2, z) \to (z_1, z_2, \zeta z)$ should be an isometry (with respect to the operator norm) for every $\zeta^k = 1$. However, simple examples show that the last one does not hold. Namely, it is not true that $\varphi_\zeta(\mathcal{R}_{II}) \subset \mathcal{R}_{II}$ for $\zeta \in \sqrt[k]{1}$.*

We might go further consider the classical Cartan domain of second type $\mathcal{R}_{II}$ in $\mathbb{C}^{\binom{n}{2}}$, here $\mathcal{R}_{II}$ is the set of all symmetric matrices of order n with the operator norm smaller than 1 (for definitions and properties see [6]), and investigate holomorphic map $\varphi : \mathcal{R}_{II} \to \mathbb{C}^{\binom{n+1}{2}}$, $\varphi((z_{jk})_{1\le j\le k\le n}) = (z_{1,1}, \ldots, z_{n,n}, z_{1,1}z_{2,2} - z_{1,2}^2, z_{1,1}z_{3,3} - z_{1,3}^2, \ldots, z_{1,1}z_{n,n} - z_{1,n}^2, \ldots, z_{n-1,n-1}z_{n,n} - z_{n-1,n}^2)$. Unfortunately, φ fails to be proper onto the image (for the same reason as φ_k are not), either. So, this indicates that there is no obvious generalization of the tetrablock in higher dimension.

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References

[1] A.A. Abouhajar, M.C. White, N.J. Young, *A Schwarz lemma for a domain related to μ-synthesis*, J. Geom. Anal., 17(2007), 717-750.

[2] S.R. Bell, *The Bergman kernel function and proper holomorphic mappings*, Trans. Amer. Math. Soc., 270(1982), 685-691.

[3] H.P. Boas, *Lu Qi-Keng's problem*, J. Korean Math. Soc., 37(2000), 253-267.

[4] H.P. Boas, S. Fu, E.J. Straube, *The Bergman kernel function: explicit formula and zeroes*, Proc. Amer. Math. Soc, 127(1999), no. 3, 805-811.

[5] A. Edigarian, Ł. Kosiński, W. Zwonek, *The Lempert Theorem and the Tetrablock*, J. Geom. Anal., 23(2013), no. 4, 1818-831.

[6] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, AMS, Providence, vol 6, 1963.

[7] M. Jarnicki, P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, Berlin, 1993.

[8] M. Jarnicki, P. Pflug, *Extension of Holomorphic Functions*, Walter de Gruyter, Berlin, 2000.

[9] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France, 109(1981), no. 4, 427-474.

[10] L. Lempert, *Holomorphic retracts and intrinsic metrics in convex domains*, Anal. Math., 8(1982),no. 4, 257-261.

[11] E. Ligocka, *On the Forelli-Rudin construction and the weighted Bergman projections*, Studia Math., 94(1989), 257-272.

[12] G. Misra, S.S. Roy and G. Zhang, *Reproducing kernel for a class of weighted Bergman spaces on the symmetrized polydisc*, Proc. Amer. Math. Soc (to appear).

[13] W. Rudin, *Proper holomorphic maps and finite reflection groups*, Indiana Univ. Math. J., 31(1982), 701-720.

[14] L. Zhang, W. Yin, *Lu Qi-Keng's problem on some complex ellipsoids*, J. Math. Anal. Appl., 357(2009), 364-370.

[15] W. Zwonek, *Geometric properties of the tetrablock*, Archiv d. Math 100(2) (2013), 159-165.