A DISC FORMULA FOR PLURISUBHARMONIC SUBEXTENTIONS AND A CHARACTERIZATION OF THINNESS OF SUBSETS OF $\mathbb{C}^n$

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Abstract. Let $X \subset \mathbb{C}^n$ be a domain and $W \subset X$ be a subdomain, $X \neq W$. Suppose that φ_1 is upper semicontinuous in $X \setminus \overline{W}$ and φ_2 is upper semicontinuous in W. We define $\varphi : X \longrightarrow \overline{\mathbb{R}}$ by $\varphi = \varphi_1$ in $X \setminus \overline{W}$, $\varphi = \min\{\varphi_1^*, \varphi_2^*\}$ on $X \cap \partial W$ and $\varphi = \varphi_2$ in W. Under suitable conditions on W and X, we will prove that

$$EH(x) = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi \circ f(e^{i\theta})d\theta, f \in \mathscr{O}(\overline{\mathbb{D}}, X), f(0) = x\right\}$$

is the largest plurisubharmonic function on X less than φ.
In case where $\varphi_1^* = \varphi_2^*$ on $X \cap \partial W$ we get the classical result of Poletsky. In case where φ_2 is big enough in W our work looks as a subextention result of Larusson-Poletsky. In some sense we have a generalization of these results. At the end we will characterize the *thinness* of a set at a point with closed analytic discs and give a version of maximum principle for certain *non-thin* sets in $\mathbb{C}^n$.

1. Introduction

The main goal of the *theory of disc functionals* is to provide disc formulas for important extremal plurisubharmonic functions in pluripotential theory, that is, to describe these functions as envelopes of disc functionals. This brings the geometry of analytic discs into pluripotential theory. Disc formulas have been proved for largest plurisubharmonic minorants in ([3], [2], [1]).

Consider a domain $X \subset \mathbb{C}^n$ and an upper semicontinuous function $\varphi : X \to \overline{\mathbb{R}}$. It is proven by different methods in [8] and [9] that,

$$\sup\{u \in PSH(X), u \leq \varphi\} = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi \circ f(e^{i\theta})d\theta, f \in \mathscr{O}(\overline{\mathbb{D}}, X), f(0) = x\right\}. \quad (1)$$

Formulas of this form are referred to as *disc formulas*. The elements of $\mathscr{O}(\overline{\mathbb{D}}, X)$ are called *analytic disc* in X. Functions from $\mathscr{O}(\overline{\mathbb{D}}, X)$ to $\overline{\mathbb{R}}$ are called *disc functionals*. We call the integral on the right hand side of (1) the *Poisson disc functional*. The envelope of a disc functional at a point x is then given by the infimum over all discs sending zero to x. In this paper we will prove a disc formula for plurisubharmonic functions. Recall that

- Edigarian in [10] showed that (1) holds if φ is plurisuperharmonic;
- Magnusson proved that (1) holds if $\varphi = \varphi_1 - \varphi_2$, where φ_1 is upper semicontinuous and φ_2 is plurisubharmonic.

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In the present paper we will prove in (section 2) that (1) holds for a function φ under the following form

$$\varphi = \begin{cases} \varphi_1 & \text{in} \quad X \setminus \overline{W}; \\ \min\{\varphi_1^*, \varphi_2^*\} & \text{on} \quad \partial W; \\ \varphi_2 & \text{in} \quad W. \end{cases}$$

Where W is a relatively compact subdomain of X and φ_1, φ_2 are upper semi continuous respectively on $X \setminus \overline{W}$ and W. The disc formula in (1) has many applications. For instance one can use it to characterize in terms of analytic discs :

- the pluripolar hull of a pluripolar set see [6];
- a non-pluripolar Borel subset of a Josefson manifold see [7];
- the polynomial hull of a compact subset of $\mathbb{C}^n$ see [8].

In fact for $K \subset \mathbb{C}^n$ compact we define the polynomial hull of K by

$$\hat{K} = \{z \in \mathbb{C}^n, |P(z)| \leq \sup_K |P| \text{ for any polynomial P}\}.$$

Let $X \subset \mathbb{C}^n$ be a Runge domain and $K \subset D$ be a compact set. In [8] it is proven that $z_0 \in \hat{K}$ if and only if for any $\epsilon > 0$ any open set V containing K there exists an analytic disc $f : \mathbb{D} \to X$ continuous on $\overline{\mathbb{D}}$ such that $f(0) = z_0$ and

$$\sigma(\{t \in \mathbb{T}, f(t) \in V\}) > 1 - \epsilon,$$

where σ is the normalized Lebesgue measure on $\mathbb{T}$.

In section 3 of this paper we give a similar characterization of the thinness of a subset of $\mathbb{C}^n$ at a given point. Our main result Corollary 6 states that :

Y is *non-thin* at x, if and only if for every $\epsilon > 0$, every neighborhood V of x and every open set U containing $Y \setminus \{x\}$ there exists $f \in \mathcal{O}(\overline{\mathbb{D}}, V)$ such that $f(0) = x$ and

$$\sigma(\mathbb{T} \cap f^{-1}(V \cap U \setminus \{x\})) > 1 - \epsilon.$$

Where $Y \subset \mathbb{C}^n$ and $x \in \mathbb{C}^n$. We will refer to (1) as Poletsky's classical theorem.

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2. Disc formula

Let X be a domain in complex affine space $\mathbb{C}^n$ and $W \subset X$ a subdomain, $X \neq W$. Consider two upper semicontinuous functions $\varphi_1 : X \setminus \overline{W} \longrightarrow \mathbb{R}$ and $\varphi_2 : W \longrightarrow \mathbb{R}$. We define $\varphi : X \longrightarrow \mathbb{R}$ by $\varphi = \varphi_1$ in $X \setminus \overline{W}$, $\varphi = \min\{\varphi_1^*, \varphi_2^*\}$ on $X \cap \partial W$ and $\varphi = \varphi_2$ in W. Notice that φ is not necessarily upper semicontinuous on X. We take the constant function $-\infty$ to be plurisubharmonic. $PSH(X)$ will be the family of plurisubharmonic functions on X and $\mathbb{D}$ denotes the unit disc, $\mathbb{T}$ the unit circle and σ the arc length measure on $\mathbb{T}$. For $x \in X$ we consider the function $EH : X \to \overline{\mathbb{R}}$ defined by

$$EH(x) = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi \circ f(e^{i\theta})d\theta, f \in \mathcal{O}(\overline{\mathbb{D}}, X), f(0) = x\right\}.$$

In this section we will prove that EH is plurisubharmonic. To do this, we will define on X an upper semicontinuous function F and prove that

$$EH(x) = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} F\circ f(e^{i\theta})d\theta, f\in\mathscr{O}(\overline{\mathbb{D}},X), f(0)=x\right\}.$$

The function F will be defined on using a subset $\mathscr{B}$ of $\mathscr{O}(\overline{\mathbb{D}},X)$. Set

$$\mathscr{B}_1 = \{f\in\mathscr{O}(\overline{\mathbb{D}},X), f(\mathbb{T})\subset W\};$$

$$\mathscr{B}_2 = \{f\in\mathscr{O}(\overline{\mathbb{D}},X), f(\mathbb{T})\subset X\setminus\overline{W}\};$$

and $\mathscr{B}=\mathscr{B}_1\cup\mathscr{B}_2$. Assume that for all $x\in X$ there is $f\in\mathscr{B}$ such that $f(0)=x$. We define by

$$F(x) = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi\circ f(e^{i\theta})d\theta, f\in\mathscr{B}, f(0)=x\right\}.$$

Proposition 1. *If for all $x\in X$ there is $f\in\mathscr{B}$ so that $f(0)=x$, then F is upper semicontinuous on X.*

Proof. Let $c\in\mathbb{R}$, $x\in X$ such that $F(x)<c$. We will prove that there is a neighborhood V of x such that $F(y)<c$ for all $y\in V$. By definition of F there is $f_0\in\mathscr{B}$ with $f_0(0)=x$ such that $\frac{1}{2\pi}\int_0^{2\pi}\varphi\circ f_0(e^{i\theta})d\theta<c$. Assume that $f_0\in\mathscr{B}_1$ then $f_0(\mathbb{T})\subset W$. As φ is upper semicontinuous on W we can find a decreasing sequence of continuous functions $(\psi_j)_j$ defined on W that converges to φ. There is $j_0>1$ such that $\frac{1}{2\pi}\int_0^{2\pi}\psi_{j_0}\circ f_0(e^{i\theta})d\theta<c$. As W is open and ψ_{j_0} is continuous then one can find $V\subset\subset X$ a small neighborhood of x such that $\{f_0(\mathbb{T})+y-x, y\in V\}\subset\subset W$, $\{f_0(\overline{\mathbb{D}})+y-x, y\in V\}\subset\subset X$ and $\frac{1}{2\pi}\int_0^{2\pi}\psi_{j_0}(f_0(e^{i\theta})+y-x)d\theta<c$ for all y. Hence $F(y)<c$ for all $y\in V$. That means, the set $\{F<c\}$ is open for all $c\in\mathbb{R}$, hence F is upper semicontinuous. □

Notice that on $X\cap\partial W$ we may have $\varphi<F$ since $F(x)=\lim_{r\to0}\sup_{B(x,r)}F$. For example take $\varphi_1=2$ and $\varphi_2=-1$. If $W\subset\subset X$ then we get $F=2$ on ∂W while $\varphi=-1$ there. Now for $x\in X$ we consider the function

$$PF(x) = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} F\circ f(e^{i\theta})d\theta, f\in\mathscr{O}(\overline{\mathbb{D}},X), f(0)=x\right\}.$$

Recall that F is upper semicontinuous, then by Poletsky's classical theorem PF is the largest plurisubharmonic function on X less than F. Our goal here is to prove that in X

$$EH = PF \le \varphi.$$

Remark that by definitions $EH\le F$ because $\mathscr{B}\subset\mathscr{O}(\overline{\mathbb{D}},X)$ and $EH\le\varphi$. If we reach to prove that $PF\le\varphi$ then we get $PF\le EH$ and other inequality will be given by Lemma 3. The following result due to Bu-Schachermayer is the core of the proof of Lemma 3. For a detailed proof of Lemma 2 see [9] *and* [1].

Lemma 2. *Let A be a compact subset of $\mathbb{T}$ and $\psi\in C(\overline{\mathbb{D}})$. Then there exists a sequence (p_k) of polynomials $p_k:\mathbb{C}\to\mathbb{C}$ satisfying*

(i) $p_k(\mathbb{D})\subset\mathbb{D}$ *and* $p_k(0)=0$;
(ii) $p_k\to0$ *uniformly on every compact subset of* $\overline{\mathbb{D}}\setminus A$ *as* $k\to\infty$;
(iii) $\int_A\psi\circ p_k(t)d\sigma(t)\longrightarrow\sigma(A)\int_{\mathbb{T}}\psi(t)d\sigma(t)$ *as* $k\to\infty$.

Lemma 3. *If for all $x\in X$ there is $f\in\mathscr{B}$ so that $f(0)=x$, then $EH\le PF$.*

Proof. Let $x \in X$, $h \in \mathscr{O}(\overline{\mathbb{D}}, X)$, with $h(0) = x$, $\epsilon > 0$ and $(\psi_j)_j$ a sequence of continuous functions decreasing to F. We will prove that there is $G_j \in \mathscr{O}(\overline{\mathbb{D}}, X)$, $G_j(0) = x$ such that

$$\int_{\mathbb{T}} \varphi \circ G_j(t) d\sigma(t) \leq \int_{\mathbb{T}} \psi_j \circ h(t) d\sigma(t) + \epsilon.$$

That means $EH(x) \leq \int_{\mathbb{T}} \psi_j \circ h(t) d\sigma(t)$ on letting j tends to ∞ we get

$$EH(x) \leq \int_{\mathbb{T}} F \circ h(t) d\sigma(t) + \epsilon.$$

On taking the infimum over all h and on letting ϵ tends to zero we get

$$EH(x) \leq PF(x).$$

Fix $x \in X$, $h \in \mathscr{O}(\overline{\mathbb{D}}, X)$, with $h(0) = x$, $\epsilon > 0$ and ψ a continuous function bigger than F. Let $t_0 \in \mathbb{T}$ there is $g_0 \in \mathscr{O}(\overline{\mathbb{D}}, \mathbb{C}^n)$ such that $g_0(0) = 0$, $h(t_0) + g_0 \in \mathscr{B}$ and

$$\int_{\mathbb{T}} \varphi(h(t_0) + g_0(z)) d\sigma(z) < \psi \circ h(t_0) + \epsilon/2.$$

We may assume that the map $h(t_0)+g_0$ belongs to $\mathscr{B}_1$. Let $U_0 \subset\subset W$ be a neighborhood of $h(t_0) + g_0(\mathbb{T})$. Take a continuous function $B_0 \in C(U_0)$ bigger than φ in U_0 such that

$$\int_{\mathbb{T}} B_0(h(t_0) + g_0(z)) d\sigma(z) < \psi \circ h(t_0) + \epsilon. \tag{2}$$

Extend B_0 to a continuous function on X. As U_0 and X are open, B_0 and ψ are continuous, then there is an open arc I_0 containing t_0 such that

$$\{h(t) + g_0(z), \quad t \in I_0, \quad z \in \overline{\mathbb{D}}\} \subset\subset X \text{ and } \{h(t) + g_0(z), \quad t \in I_0, \quad z \in \mathbb{T}\} \subset\subset U_0;$$
$$|B_0(h(t) + g_0(z)) - B_0(h(t_0) + g_0(z))| < \epsilon \text{ for } t \in I_0, \quad z \in \mathbb{T};$$
$$|\psi \circ h(t) - \psi \circ h(t_0)| < \epsilon, \text{ for } t \in I_0.$$

By compactness there is $N > 0$, points $t_1, \cdots, t_N \in \mathbb{T}$, open arcs $I_1, \cdots, I_N$, holomorphic maps $g_1, \cdots, g_N \in \mathscr{O}(\overline{\mathbb{D}}, \mathbb{C}^n)$, open sets $U_1 \cdots, U_N$ relatively compact either in W or in $X \setminus \overline{W}$ and $B_1, \cdots, B_N$ continuous functions on X with $\varphi \leq B_j$ on U_j for $j = 1, \cdots, N$ such that

$$t_j \in I_j, \quad g_j(0) = 0, \quad h(t_j) + g_j \in \mathscr{B} \text{ and } \mathbb{T} \subset \cup I_j;$$
$$\{h(t) + g_j(z), \quad t \in I_j, \quad z \in \overline{\mathbb{D}}\} \subset\subset X \text{ and } \{h(t) + g_j(z), \quad t \in I_j, \quad z \in \mathbb{T}\} \subset\subset U_j;$$

$$\int_{\mathbb{T}} B_j(h(t_j) + g_j(z)) d\sigma(z) < \psi \circ h(t_j) + \epsilon; \tag{3}$$

$$|B_j(h(t) + g_0(z)) - B_j(h(t_j) + g_0(z))| < \epsilon \text{ for } t \in I_j, \quad z \in \mathbb{T}; \tag{4}$$

$$|\psi \circ h(t) - \psi \circ h(t_j)| < \epsilon, \text{ for } t \in I_j. \tag{5}$$

Choose δ_0 very small such that for all j

$$\left\{h(t) + g_j(z) + x, \quad ||x|| < \delta_0, \quad t \in I_j, \quad z \in \overline{\mathbb{D}}\right\} \subset\subset X;$$
$$\{h(t) + g_j(z) + x, \quad ||x|| < \delta_0, \quad t \in I_j, \quad z \in \mathbb{T}\} \subset\subset U_j$$

and $K \subset\subset X$ an open set containing

$$\bigcup_{j=1}^{N} \left\{h(t) + g_j(z) + x, \quad ||x|| < \delta_0, \quad t \in I_j, \quad z \in \overline{\mathbb{D}}\right\} \cup h(\overline{\mathbb{D}}).$$

Take $C > \sum_j \sup_{\overline{K}} |B_j| + \sup_{\overline{K}} |\psi| + |\sup_{\overline{K}} \varphi|$ and disjoint closed arcs $J_j \subset I_j$ such that

$$C\sigma(\mathbb{T} \setminus \cup_j J_j) < \epsilon. \tag{6}$$

By uniform continuity of B_j on $\overline{K}$ there is $0 < \delta < \delta_0$ such that

$$|B_j(x_1) - B_j(x_2)| < \epsilon, \text{ for all } x_1, \quad x_2 \in K \text{ with } ||x_1 - x_2|| < \delta, \text{ for all } j. \tag{7}$$

Take a small open neighborhood V_j of J_j such that

$$(\cup_{i=1, i\neq j}^{N} J_i) \cup \{0\} \subset \mathbb{C} \setminus V_j.$$

Set $K_j = \overline{\mathbb{D}} \setminus V_j$. By Lemma 2 for each $i = 1, \cdots, N$ there is a polynomial p_i such that:

- $p_i(0) = 0$ and $p_i(\mathbb{D}) \subset \mathbb{D}$;
- $||g_i \circ p_i(z)|| < \delta/N$ for all $z \in K_i$;
- $\int_{J_i} B_i(h(t_i) + g_i \circ p_i(t))d\sigma(t) < \sigma(J_i) \int_{\mathbb{T}} B_i(h(t_i) + g_i(t))d\sigma(t) + \epsilon/N$.

Set

$$G(z) = h(z) + \sum_{i=1}^{N} g_i \circ p_i(z) \text{ for all } z \in \overline{\mathbb{D}}.$$

Then $G \in \mathscr{O}(\overline{\mathbb{D}}, K)$ and $G(0) = h(0)$ and we have

$$\begin{aligned}
\int_{\mathbb{T}} \varphi \circ G(t) d\sigma(t) &\leq \sum_{i=1}^{N} \int_{J_i} \varphi \circ G(t) d\sigma(t) + \epsilon \qquad \text{"by (6)} \\
&= \sum_{i=1}^{N} \int_{J_i} \varphi \left(h(t) + g_i \circ p_i(t) + \sum_{j=1, i\neq j}^{N} g_j \circ p_j(t) \right) d\sigma(t) + \epsilon \\
&\leq \sum_{i=1}^{N} \int_{J_i} B_i \left(h(t) + g_i \circ p_i(t) + \sum_{j=1, i\neq j}^{N} g_j \circ p_j(t) \right) d\sigma(t) + \epsilon \\
&\leq \sum_{i=1}^{N} \int_{J_i} B_i \left(h(t) + g_i \circ p_i(t) \right) d\sigma(t) + 2\epsilon \qquad \text{"by of (7)} \\
&\leq \sum_{i=1}^{N} \int_{J_i} B_i \left(h(t_i) + g_i \circ p_i(t) \right) d\sigma(t) + 3\epsilon \qquad \text{" by (4)} \\
&\leq \sum_{i=1}^{N} \sigma(J_i) \int_{\mathbb{T}} B_i \left(h(t_i) + g_i(t) \right) d\sigma(t) + 4\epsilon \qquad \text{"by Lemma 2} \\
&\leq \sum_{i=1}^{N} \sigma(J_i) \psi \circ h(t_i) + 5\epsilon \qquad \text{"by (3),,} \\
&\leq \sum_{i=1}^{N} \int_{J_i} \psi \circ h(t) d\sigma(t) + 6\epsilon \qquad \text{"by (5)} \\
&\leq \int_{\mathbb{T}} \psi \circ h(t) d\sigma(t) + 7\epsilon \qquad \text{by (6).}
\end{aligned}$$

We get $EH(x) \leq \int_{\mathbb{T}} \psi \circ h(t) d\sigma(t) + 7\epsilon$ for all $\epsilon > 0$, continuous function $\psi \geq F$ and $h \in \mathscr{O}(\overline{\mathbb{D}}, X)$ with $h(0) = x$. Hence $EH(x) \leq PF(x)$. □

Consider $v : X \to \mathbb{R}$. Recall that

$$\limsup_{y\to x} v(y) = \lim_{r\to 0} \sup\{v(y), \quad y \in B(x,r)\}, \quad x \in X.$$

$$\limsup_{y\to x,\, y\in Y} v(y) = \lim_{r\to 0} \sup\{v(y), \quad y \in B(x,r)\cap Y\}, \quad x \in \overline{Y}.$$

In what follows the word thin means $\mathbb{C}^n$-thin.

Definition 1. Let Y be a subset of $\mathbb{C}^n$ and $x \in \mathbb{C}^n$. Then Y is *non-thin* at x if $x \in \overline{Y \setminus \{x\}}$ and if, for every plurisubharmonic function u defined on a neighborhood of x one has

$$\limsup_{z\to x,\ z\in Y\setminus\{x\}} u(z) = u(x).$$

As example we have, a connected set containing more than one point is non-thin at every point of its closure see Theorem 3.8.3 in [4]. *If $h \in \mathscr{O}(\overline{\mathbb{D}}, \mathbb{C}^n)$ then the set $h([0,1])$ is not thin at any of its points see Corollary 4.8.5 in* [5].

Theorem 4. *Let $X \subset \mathbb{C}^n$ be a domain and $W \subset X$. Suppose that*

i) $\{x \in X,$ *there is* $f \in \mathscr{B}$ *such that* $f(0) = x\} = X$;

ii) $X \setminus \overline{W}$ *and* W *are subdomains of* X.

Then $EH \in PSH(X)$ and

$$\sup\{v(x), v \in PSH(X), v \le \varphi\} = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi\circ f(e^{i\theta})d\theta, f \in \mathscr{O}(\overline{\mathbb{D}}, X), f(0) = x\right\}.$$

Proof. The condition $i)$ allows us to define F. Condition $ii)$ ensures that the sets W and $X \setminus \overline{W}$ are non-thin at points of ∂W. We have

$$PF \le EH \le PF \le \varphi.$$

Indeed, the third inequality holds because, we have $PF \le F \le \varphi$ in $X \setminus \partial W$ (because of constant maps in $\mathscr{B}$). Let $x \in X \cap \partial W$, we may assume that $\varphi(x) = \varphi_2^*(x)$. As $PF \in PSH(X)$ and W is *non-thin* at x then

$$PF(x) = \limsup_{z\to x,\, z\in W} PF(z) \le \limsup_{z\to x,\, z\in W} \varphi_2(z) = \varphi_2^*(x) = \varphi(x).$$

This for all $x \in \partial W$. Thus $PF \le \varphi$ on X. The second holds because of Lemma 3. The first one holds because $PF \in PSH(X)$ and $PF \le \varphi$. Hence $PF = EH \in PSH(X)$. Obviously for all $u \in PSH(X)$, $u \le \varphi$ we have $u \le EH$ hence $\sup\{v, v \in PSH(X), v \le \varphi\} \le EH$. As $EH \in PSH(X)$ and less than φ then we have equality. □

As φ may be, not upper semicontinuous then our formula generalizes Poletsky's classical formula . For properties of thin sets we refer to [4] *and* [5].

3. Thinness

Let u be a function plurisubharmonic on a neighborhood of $z_0 \in \mathbb{C}^n$. Even though u may be discontinuous at z_0, it is still always true that

$$\limsup_{z\to z_0} u(z) = u(z_0).$$

By upper semicontinuity, we certainly have $\limsup_{z\to z_0} u(z) \le u(z_0)$, and if the inequality were strict, then u would violate the submean inequality on a neighborhood of z_0. The *situation may change if we take the limit along a set U whose closure contains z_0. For instance let $(z_n)_{n>1}$ be a sequence converging to z_0 and $U = \{z_1, z_2, \cdots\}$. It is easy to find $u \in PSH(\mathbb{C}^n)$ such that*

$$\limsup_{z\to z_0, z\in U} u(z) < u(z_0).$$

We say that U is thin at z_0. In this section we will give a characterization of thinness of a set at a given point in $\mathbb{C}^n$ in terms of analytic discs.

Theorem 5. *Let $U \subset \mathbb{C}^n$ be open and $x \in \mathbb{C}^n$. Then the two conditions are equivalent:*

i) *U is non-thin at x;*

ii) *For all $\epsilon > 0$, neighborhood V of x there is $f \in \mathscr{O}(\overline{\mathbb{D}}, V)$ such that $f(0) = x$ and*

$$\sigma(\mathbb{T} \cap f^{-1}(V \cap U \setminus \{x\})) > 1 - \epsilon.$$

Proof. Assume *i)*. Let $\epsilon > 0$ and V a neighborhood of x. Let $V_1 \subset\subset V$ be a small ball centered at x. Then by Poletsky's classical theorem the function $u_{(U\setminus\{x\})\cap V_1, V_1}$ is plurisubharmonic in V_1, where

$$u_{(U\setminus\{x\})\cap V_1,V_1}(x) = \inf\{-\sigma(\mathbb{T} \cap f^{-1}((U \setminus \{x\}) \cap V_1)), f \in \mathscr{O}(\overline{\mathbb{D}}, V_1), f(0) = x\}.$$

Since U is *non-thin* at x then

$$u_{(U\setminus\{x\})\cap V_1,V_1}(x) = \limsup_{z\to x,\ z\in U\setminus\{x\}} u_{(U\setminus\{x\})\cap V_1,V_1}(z) = -1,$$

thus there is $f \in \mathscr{O}(\overline{\mathbb{D}}, V)$ such that $f(0) = x$ and

$$-\sigma(\mathbb{T} \cap f^{-1}(V \cap U \setminus \{x\})) < -1 + \epsilon.$$

Suppose *ii)*. Let $r_0 > 0$ and $u \in PSH(B(x, r_0))$. For any $0 < r < r_0$ we set

$$c_r = \sup\{u(z), z \in \overline{B}(x, r) \cap U \setminus \{x\}\}.$$

Take $M > |\sup_{\overline{B}(x,r)} u|$. By *ii)* we have, for any $\epsilon > 0$ there is $f_\epsilon \in \mathscr{O}(\overline{\mathbb{D}}, B(x, r))$ with $f_\epsilon(0) = x$ such that

$$\sigma(\mathbb{T} \cap f_\epsilon^{-1}(B(x, r) \cap U \setminus \{x\})) > 1 - \epsilon.$$

Set $A = \mathbb{T} \setminus (\mathbb{T} \cap f_\epsilon^{-1}((U \setminus \{x\}) \cap B(x, r)))$ thus we have

$$\begin{aligned} u(x) \le \int_{\mathbb{T}} u \circ f_\epsilon(t) d\sigma(t) &\le \int_{\mathbb{T}\setminus A} u \circ f_\epsilon(t) d\sigma(t) + \int_A u \circ f_\epsilon(t) d\sigma(t) \\ &\le c_r(1 - \sigma(A)) + M\sigma(A) \le c_r + (|c_r| + M)\epsilon. \end{aligned}$$

This for all $\epsilon > 0$, hence when $\epsilon \to 0$ we get $u(x) \le c_r$. As r was taken arbitrarily then we have

$$u(x) \le \inf_{r>0} c_r = \limsup_{z\to x,\ z\in U\setminus\{x\}} u(z) \le \limsup_{z\to x} u(z) = u(x).$$

This for all u plurisubharmonic in a neighborhood of x. Hence U is *non-thin* at x. □

In the light of Corollary 4.8.3 in [5] *we have the following.*

Corollary 6. *Let $Y \subset \mathbb{C}^n$ and $x \in \mathbb{C}^n$. Then the following conditions are equivalent*

i) *Y is non-thin at x;*

ii) *For every $\epsilon > 0$, neighborhood V of x and every open set U containing $Y \setminus \{x\}$ there exists $f \in \mathscr{O}(\overline{\mathbb{D}}, V)$ such that $f(0) = x$ and*

$$\sigma(\mathbb{T} \cap f^{-1}(V \cap U \setminus \{x\})) > 1 - \epsilon.$$

For an open set $X \subset \mathbb{C}^n$, $u \in PSH(X)$ and $K \subset X$ compact it is well known (by the classical maximum principle) that $\sup_K u = \sup_{\partial K} u$. *We will state a similar result for certain subsets of X not necessarily compact.*

Theorem 7. *Let $U \subset\subset X \subset \mathbb{C}^n$, where X is open. If U is non-thin at every point of its closure, then for all $u \in PSH(X)$ one has*

$$\sup_U u = \sup_{\partial U} u.$$

Proof. Let U be *non-thin* at every point of its closure. By the classical maximum principle we have

$$\sup_U u \leq \sup_{\overline{U}} u = \sup_{\partial \overline{U}} u \leq \sup_{\partial U} u.$$

Let $x \in \partial U$ and $r > 0$ small so that $\overline{B}(x, r) \subset X$. As U is *non-thin* at x then

$$u(x) = \inf_{\rho > 0} \sup_{U \cap \overline{B}(x,\rho)} u \leq \sup_{U \cap \overline{B}(x,r)} u \leq \sup_U u.$$

This for all $x \in \partial U$. Hence $\sup_{\partial U} u \leq \sup_U u$. □

Corollary 8. *Let $X \subset \mathbb{C}^n$ be open and $U \subset\subset X$ be connected and $u \in PSH(X)$, then*

$$\sup_U u = \sup_{\partial U} u.$$

Corollary 9. *Let $X \subset \mathbb{C}^n$ be open and let $U \subset\subset X$ be open. If U is non-thin at every point of its closure, then*

$$u_{U,X} = u_{\overline{U},X}.$$

Moreover if X is hyperconvex, then $u_{U,X}$ is continuous in X and $\lim_{z \to \partial X} u_{U,X}(z) = 0$.

Proof. Recall that

$$u_{U,X} = \sup\{v \in PSH(X), v < 0, v|U \leq -1\};$$

$$u_{\overline{U},X} = \sup\{v \in PSH(X) v < 0, v|\overline{U} \leq -1\}.$$

Notice that for all $v \in PSH(X)^-$ with $v \leq -1$ on $\overline{U}$. We have $v \leq u_{U,X}$. Thus

$$u_{\overline{U},X} \leq u_{U,X}.$$

As $u_{U,X} \in PSH(X)^-$ and $u_{U,X} = -1$ in U, then by Theorem 7, $u_{U,X} = -1$ on $\overline{U}$. Hence $u_{U,X}$ is in the family defining $u_{\overline{U},X}$. Thus $u_{U,X} \leq u_{\overline{U},X}$.
If X is hyperconvex then by Proposition 4.5.3 in [5], $u_{\overline{U},X}$ is continuous on X, hence $u_{U,X}$ is continuous and by Proposition 4.5.2 we have $\lim_{z \to \partial X} u_{U,X}(z) = 0$. □

References

[1] Magnússon, B. S. *Extremal ω-plurisubharmonic functions as envelopes of disc functionals.* Arkiv Math. **49** (2011) 383–399.

[2] Poletsky, E. A. *Plurisubharmonic functions as solutions of variational problems.* Several complex variables and complex geometry (Santa Cruz CA, 1989), 163–171. Proc. Sympos. Pure Math., 52, Part 1. Amer. Math. Soc., 1991.

[3] Lárusson, F. and Poletsky, E. A. *Plurisubharmonic subextentions as envelopes of disc functionals.* Michigan. Math. J. **62** (2013), 551-565.

[4] Thomas Ransford; *Potential Theory in the Complex Plane.* London Mathematical Society Student Texts 28, Cambridge university press 1995.

[5] M. Klimek, *Pluripotential Theory*, London Math. Soc., Monographs New Series 6, Clarendon Press, Oxford, 1991.
[6] N. Levenberg and E. A. Poletsky, *Pluripolar Hulls*, Michigan Math. J. **46** (1999) 151–162
[7] A. Edigarian and R. Sigurdsson, *The Relative Extremal Function for Borel sets in Complex Manifolds*, (2006) arXiv:math/0607313v1 [math.CV].
[8] E. A. Poletsky, *Holomorphic currents*, Indiana Univ. Math. J., **42** (1993), 85–144.
[9] S. Q. Bu and W. Schachermayer, Approximation of Jensen measures by image measures under holomorphic functions and applications, Trans. Amer. Math. Soc., 331 (1992), pp. 585-608.
[10] A. Edigarian, A note on Larusson-Sigurdsson's paper, Math. Scand, 92 (2003), pp. 309-319.

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