THE KOBAYASHI BALLS OF ($\mathbb{C}$-)CONVEX DOMAINS

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Abstract. A pure geometric description of the Kobayashi balls of ($\mathbb{C}$-)convex domains is given in terms of the so-called minimal basis.

1. Introduction and results

Let D be a domain in $\mathbb{C}^n$. Denote by c_D and l_D the Carathéodory distance and the Lempert function of D, respectively:

$$c_D(z,w) = \sup\{\tanh^{-1}|f(w)| : f \in \mathcal{O}(D,\mathbb{D}),\ \text{ with } f(z)=0\},$$

$$l_D(z,w) = \inf\{\tanh^{-1}|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D},D) \text{ with } \varphi(0)=z, \varphi(\alpha)=w\},$$

where $\mathbb{D}$ is the unit disc. The Kobayashi distance k_D is the largest pseudodistance not exceeding l_D.

We are interested in a description of the Kobayashi balls near boundary points of convex and, more generally, $\mathbb{C}$-convex domains in terms of parameters that reflect the geometry of the boundary. Such a description is

The first results in this direction can be found in [2, Theorems 1 and 5.1], where the strongly pseudoconvex case in $\mathbb{C}^n$ and the weakly pseudoconvex finite type case in $\mathbb{C}^2$ are discussed with applications[1] to invariant forms of the Fatou type theorems (for the boundary values). The weakly pseudoconvex finite type case in $\mathbb{C}^2$, as well as the convex finite type case in $\mathbb{C}^n$, are treated in [6, Propositions 8.8 and 8.9] as byproducts of long considerations. The strongly pseudoconvex case in $\mathbb{C}^n$ and the weakly pseudoconvex finite type in $\mathbb{C}^2$ are particular cases of the pseudoconvex Levi corank one case which is considered in [3, Theorem 1.3]. The behavior of the Kobayashi balls in all the mentioned

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[1] See also [14] for complex ellipsoids.

results is given in terms of the Levi geometry of the boundary which is assumed smooth and bounded.

Our aim is to describe the Kobayashi balls of ($\mathbb{C}$-)convex domains (not necessarily smooth and bounded) in terms of the so-called minimal basis (cf. [4, 9, 12]. The constants that appear depend only on the radius of the balls and the dimension of the domains. The respective proof is short and pure geometric. The obtained result covers [6, Propositions 8.8 and 8.9].

Assume that D contains no complex lines. Let $q \in D$ and $d_D(q) = \operatorname{dist}(q, \partial D)$. Choose $q^1 \in \partial D$ so that $\tau_1(q) := \|q^1 - q\| = d_D(q)$. Put $H_1 = q + \operatorname{span}(q^1 - q)^\perp$ and $D_1 = D \cap H_1$. Let $q^2 \in \partial D_1$ so that $\tau_2(q) := \|q^2 - q\| = d_{D_1}(q)$. Put $H_2 = q + \operatorname{span}(q^1 - q, q^2 - q)^\perp$, $D_2 = D \cap H_2$ and so on. Thus we get an orthonormal basis of the vectors $e_j = \dfrac{q^j - q}{\|q^j - q\|}$, $1 \le j \le n$, which is called minimal for D at q, and positive numbers $\tau_1(q) \le \tau_2(q) \le \dots \le \tau_n(q)$ (the basis and the numbers are not uniquely determined). After rotation we may assume that $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{C}^n$.

Recall now that a open set D in $\mathbb{C}^n$ is said to be (cf. [1]):

• $\mathbb{C}$-convex if any non-empty intersection with a complex line is a simply connected domain.

• linearly (weakly linearly convex) convex if for any $p \in \mathbb{C}^n \setminus D$ ($p \in \partial D$) there exists a complex hyperplane through a which does not intersect D.

Note that convexity $\Rightarrow$ $\mathbb{C}$-convexity $\Rightarrow$ linear convexity $\Rightarrow$ weak linear convexity (cf. [1, Theorem 2.3.9 ii)] for the second implication). Moreover, in the case of C^1-smooth bounded domains the last three notions coincide (cf. [1, Corollary 2.5.6].

In view of this remark and the inequalities $c_D \le k_D \le l_D$, we have the following quantitative information about the Carathéodory/Kobayashi/ Lempert balls of ($\mathbb{C}$-)convex domains.[2]

Theorem 1. *Let D be a domain in $\mathbb{C}^n$, containing no complex lines, and $q \in D$. Assume that the standard basis of $\mathbb{C}^n$ is minimal for D at q. Let $r > 0$.*

(i) If D is weakly linearly convex, then

$$\max_{1\le j\le n} \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r}-1}{n(e^{2r}+1)} \Rightarrow \sum_{j=1}^n \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r}-1}{e^{2r}+1}$$
$$\Rightarrow z \in D \text{ and } l_D(q,z) < r.$$

[2]By the Lempert theorem, $c_D = k_D = l_D$ in the convex case, as well as in the bounded C^2-smooth $\mathbb{C}^2$-convex case (cf. [11]).

(ii) If D is convex, then $c_D(q,z)<r$ implies $\max\limits_{1\le j\le n}\dfrac{|z_j-q_j|}{\tau_j(q)}<e^{2r}-1$.

(iii) If D is $\mathbb{C}$-convex, then $c_D(q,z)<r$ implies $\max\limits_{1\le j\le n}\dfrac{|z_j-q_j|}{\tau_j(q)}<e^{4r}-1$.

So there exist constants $c'=c'(r,n)$ and $c''=c''(r)$ such that

$$\mathbb{D}(q_1,c'\tau_1(q))\times\cdots\times\mathbb{D}(q_n,c'\tau_n(q))\subset \mathrm{kob}_D(q,r)$$
$$\subset \mathbb{D}(q_1,c''\tau_1(q))\times\cdots\times\mathbb{D}(q_n,c''\tau_n(q)),$$

where $\mathrm{kob}_D(q,r)$ is the Kobayashi ball $\{z\in D : k(q,z)<r\}$ and $\mathbb{D}(p,r)=\{z\in\mathbb{C}:|z-p|<r\}$. By [4, Lemma 3.10], the sizes of these polydiscs are comparable (in terms of small/big constant depending on D) with the sizes of polydiscs in [3, 6] arising from the Levi geometry of the boundary. Thus Proposition 1 extends [6, Propositions 8.9].

Note also that if D is a proper $\mathbb{C}$-convex domain in $\mathbb{C}^n$ containing complex line, then it is biholomorphic to $D'\times\mathbb{C}^{n-k}$, where D' is a bounded domain in $\mathbb{C}^k$, $0<k<n$. (cf. Proposition 3 and the preceding remark in [10]). So $\tau_k(q)<\infty=\tau_{k+1}(q)$ and it is easy to see that Theorem 1 remains true.

To prove Theorem 1, we need the planar cases of following

Proposition 2. *(i) Let D be proper convex domain in $\mathbb{C}^n$. Then (cf. [13, (2)])*

$$c_D(z,w)\ge \frac{1}{2}\log\frac{d_D(z)}{d_D(w)}.$$

Moreover, if $n=1$, then

$$c_D(z,w)\ge \frac{1}{2}\log\left(1+\frac{|z-w|}{d_D(w)}\right).$$

(ii) Let D be proper $\mathbb{C}$-convex domain in $\mathbb{C}^n$. Then

$$c_D(z,w)\ge \frac{1}{4}\log\frac{d_D(z)}{d_D(w)}.$$

Moreover, if $n=1$, then

$$c_D(z,w)\ge \frac{1}{4}\log\left(1+\frac{|z-w|}{d_D(w)}\right).$$

The constants $1/2$ and $1/4$ are sharp as the examples $D=\mathbb{D}$ and $D=\mathbb{C}_*\setminus\mathbb{R}^+$ show. Note that in the $\mathbb{C}$-convex case the weaker estimate

$$c_D(z,w)\ge \frac{1}{4}\log\frac{d_D(z)}{4d_D(w)}$$

is contained in [13, Proposition 2]

Theorem 1 has a local version.

Proposition 3. *Let D be a domain in $\mathbb{C}^n$ whose boundary contains no affine discs through $a \in \partial D$. Assume that the standard basis of $\mathbb{C}^n$ is minimal for D at $q \in D$. Let $r > r' > 0$.*

(i) If D is weakly linearly convex near a, then

$$\max_{1\le j\le n} \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r}-1}{n(e^{2r}+1)} \Rightarrow \sum_{j=1}^n \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r}-1}{e^{2r}+1}$$
$$\Rightarrow z \in D \text{ and } l_D(q,z) < r.$$

for q sufficiently close to a.

(ii) If D is convex near a, then $k_D(q,z) < r'$ implies $\max_{1\le j\le n} \frac{|z_j - q_j|}{\tau_j(q)} <$ $e^{2r}-1$ for q sufficiently close to a.

(iii) If D is $\mathbb{C}$-convex near a and bounded, then $l_D(q,z) < r'$ implies $\max_{1\le j\le n} \frac{|z_j - q_j|}{\tau_j(q)} < e^{4r}-1$ for q sufficiently close to a.

By any of the above three notions of convexity near a we mean that there exists a neighborhood U of a such that $D \cap U$ is an open set with the respective global convexity.

Note that in the convex case, as well as in the C^1-smooth $\mathbb{C}$-convex case, if ∂D contains no affine discs through a, then ∂D contains no analytic discs through a (cf. [12, Propoisition 7]).

2. Proofs

Proof of Theorem 1. (i) Since D contains the discs $\mathbb{D}(q_1, \tau_1(q)), \dots, \mathbb{D}(q_n, \tau_n(q))$ (lying in the respective coordinate complex planes), it contains their convex hull

$$C = \{\zeta \in \mathbb{C}^n : h(\zeta) = \sum_{j=1}^n \frac{|\zeta_j - q_j|}{\tau_j(q)} < 1\}$$

(cf. [12, Lemma 15]). Then

$$l_D(q,z) \le l_C(q,z) = \tanh^{-1} h(z)$$

(cf. [5, Proposition 3.1.10]) which implies (i).

Before proving (ii) and (iii) note that by ($\mathbb{C}$-)convexity and the construction of the minimal basis there exists a complex hyperplane $q^{j+1} + W_j$ through q^{j+1} that is disjoint from D, $j = 0, \dots, n-1$. It is not difficult to see that W_j is given by the equation

$$\alpha_{j,1}\zeta_1 + \dots + \alpha_{j,j}\zeta_j + \zeta_{j+1} = 0.$$

Let $\Lambda : \mathbb{C}^n \to \mathbb{C}^n$ be the linear mapping with matrix whose rows are given by the vectors $(\alpha_{j,1}, \dots, \alpha_{j,j}, 1, 0, \dots, 0)$. Set $\Lambda_q(\zeta) = q + \Lambda(\zeta - q)$. Note that $G = \Lambda_q(D)$ is a ($\mathbb{C}$-)convex domain. Denote by G_j the projection of G onto j-th coordinate plane. Then $G \subset G' = G_1 \times \dots \times G_n$ and the product formula for the Carathéodory distance (cf. [5, Theorem 9.5]) implies that

$$c_D(q, z) \ge c_{G'}(q, \Lambda_q(z)) = \max_{1 \le j \le n} c_{G_j}(q_j, z_j). \tag{1}$$

Observe also that $d_{G_j}(q_j) = \tau_j(q)$.

(ii) If D is a convex domain, then G_j is a convex domain. Hence, by Proposition 2 (i),

$$c_{G_j}(q_j, z_j) \ge \frac{1}{2} \log \left(1 + \frac{|z_j - q_j|}{\tau_j(q)} \right)$$

and (ii) follows from here and (1).

(iii) If D is a $\mathbb{C}$-convex domain, then G_j is a simple connected domain (cf. [1, Theorem 2.3.6]). Hence, by Proposition 2 (ii),

$$c_{G_j}(q_j, z_j) \ge \frac{1}{4} \log \left(1 + \frac{|z_j - q_j|}{\tau_j(q)} \right)$$

and (iii) follows from here and (1).

Proof of Proposition 2. After translation and rotation, we may assume that $0 \in \partial D$ and $w = (d_D(w), 0, \dots, 0)$.

(i) We have that $D \subset \Pi^+ = \{\zeta \in \mathbb{C}^n : \mathrm{Re}\, \zeta_1 > 0\}$ and hence

$$c_D(z, w) \ge c_{\Pi^+}(z, w) = \tanh^{-1} \left| \frac{z_1 - w_1}{z_1 + \overline{w}_1} \right|$$

$$\ge \tanh^{-1} \frac{|z_1 - w_1|}{|z_1 - w_1| + 2d_D(w)} = \frac{1}{2} \log \left(1 + \frac{|z_1 - w_1|}{d_D(w)} \right).$$

(ii) It follows by weak linear convexity that $D \cap \{\zeta_1 \in \mathbb{C}^n : \zeta_1 = 0\} = \varnothing$. Denote by D_1 the projection of D onto the ζ_1-plane. Let γ_G the Carathéodory metric of a domain G in $\mathbb{C}^k$:

$$\gamma_G(\zeta; X) = \sup\{|f'(\zeta)X| : f \in \mathcal{O}(G, \mathbb{D})\}, \quad \zeta \in G,\ X \in \mathbb{C}^k.$$

The Köbe 1/4 theorem implies that

$$\gamma_{D_1}(\zeta_1; e_1) \ge \frac{1}{4d_{D_1}(\zeta_1)} \ge \frac{1}{4|\zeta_1|}.$$

Since D_1 is a simply connected domain (cf. [1, Theorem 2.3.6]), then

$$c_D(z, w) \ge c_{D_1}(z_1, w_1) = \inf_s \int_0^1 \gamma_{D_1}(s(t); s'(t)dt \ge \frac{1}{4} \inf_s \int_0^1 \left| \frac{s'(t)}{s(t)} \right| dt,$$

where the infimum is taken over all smooth curves $s : [0, 1] \to D_1$ with $s(0) = z_1$ and $s(1) = w_1$ (cf. [5]).

Set now

$$d(\zeta_1, \eta_1) = \log \max(1 + |1 - \zeta_1/\eta_1|, 1 + |1 - \eta_1/\zeta_1|).$$

It is easy to check that d is a distance on $\mathbb{C}_*$[3] with "derivative"

$$\lim_{\lambda \to 0} \frac{d(\zeta_1, \zeta_1 + \lambda)}{|\lambda|} = \frac{1}{|\zeta_1|}.$$

Then (cf. [5, Lemma 4.3.3) (d)])

$$\inf_s \int_0^1 \left| \frac{s'(t)}{s(t)} \right| dt \geq d(z_1, w_1)$$

and hence

$$c_D(z, w) \geq \frac{1}{4} d(z_1, w_1) \geq \frac{1}{4} \log \left(1 + \frac{|z_1 - w_1|}{d_D(w)} \right).$$

Proof of Proposition 3. (i) Using Theorem 1(i), it is enough to show that $\lim_{q \to a} \tau_n(q) = 0$. Assume the contrary. Then there exists a sequence of points $(q^j) \to a$ such that $(\tau_n(q^j)) \to \varepsilon > 0$ and $(e^j) \to e$, where e^j is the last vector of the minimal basis for D at q^j. We may find a bounded neighborhood U of a such that $D \cap U$ is a weakly linearly convex open set. Shrinking ε (if necessary), it follows that the e-directional disc Δ with center q and radius ε is a limit of affine discs in $D \cap U$. Since $D \cap U$ is a taut open set (cf. [11, Proposition 1.5]), then $\Delta \subset \partial D$, a contradiction.

(ii) Having in mind Theorem 1 (ii), it is enough to show the following.

Claim 1. Let U be a neighborhood of a such that $D \cap U$ is convex. There exist neighborhoods $W \subset V \subset U$ of a such that if $q \in D \cap W$ and $k_D(q, z) < r'$, then $z \in V$ and $pk_{D \cap U}(q, z) \leq k_D(q, z)$, where $p = r'/r$.

To prove this claim, recall that k_D is the integrated form of the Kobayashi metric

$$\kappa_D(\zeta; X) = \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z, \alpha\varphi'(0) = X\}$$

[3]Let $a, b, c \in \mathbb{C}_*$ and $d_1 = 1 - a/b$, $d_2 = 1 - b/c$, $d_3 = 1 - a/c$. We may assume that $d(a, c) = \log(1 + |d_3|)$. Then

$$d(a, b) + d(b, c) \geq \log(1 + |d_1|) + \log(1 + |d_2|)$$

$$= \log(1 + |d_1| + |d_2| + |d_3 - d_2 - d_1|) \geq \log(1 + |d_3|) = d(a, c).$$

(cf. [5]). Fix an $\varepsilon > 0$. Then we may find a smooth curve $s : [0,1] \to D$ such that $s(0) = q$, $s(1) = z$ and

$$k_D(q,z) + \varepsilon > \int_0^1 \kappa_D(s(t); s'(t))dt.$$

Since $D \cap U$ is convex and its boundary contains no affine discs through a, then a is a peak point for $D \cap U$ (cf. [8, Theorem 6]). Hence the strong localization property for the Kobayashi metric holds (cf. [7, Theorem 1 and Corollary 2]). So there exists a neighborhood $V \subset U$ of a such that

$$\kappa_D(\zeta; X) \geq p\kappa_{D\cap U}(\zeta; X), \quad \zeta \in D \cap V,\ X \in \mathbb{C}^n.$$

Set $t' = \sup\{t : s([0,t]) \subset V\}$ and $z' = s(t')$. Then

$$\begin{aligned} r' + \varepsilon > k_D(q,z) + \varepsilon &> \int_0^{t'} \kappa_D(s(t); s'(t))dt \\ &\geq p \int_0^{t'} \kappa_{D\cap U}(s(t); s'(t))dt \geq pk_{D\cap U}(q,z') \geq pc_{D\cap U}(q,z'). \end{aligned}$$

Taking a peak function for $D\cap U$ at a as a competitor in the definition of $c_{D\cap U}$, it follows that

$$\lim_{q\to a} \inf_{\zeta \notin V} c_{D\cap U}(q,\zeta) = +\infty.$$

Therefore, we may find a neighborhood $W \subset V$ such that if $q \in W$, then $z' \in V$. Therefore $z' = z$ and the claim follows by letting $\varepsilon \to 0$.

(iii) Note the proof of [11, Proposition 1.5] implies the tautness of $\mathbb{C}$-convex domains. Then, in view of Theorem 1 (iii), it suffices to show the following.

Claim 2. Let U be a neighborhood of a such that $D \cap U$ is taut. There exist neighborhoods $W \subset V \subset U$ of a such that if $q \in D\cap W$ and $l_D(q,z) < r'$, then $z \in V$ and $pl_{D\cap U}(q,z) \leq l_D(q,z)$, where $p = r'/r$.

We point out that, in contrast to (ii), we do not know if a is a local peak point.

It is easy to see that Claim 2 will be a consequence of

Claim 2'. If $(\varphi_j) \subset \mathcal{O}(\mathbb{D}, D)$ and $\varphi_j(0) \to a$, then $\varphi_j \rightrightarrows a$.

To prove Claim 2', assume the contrary. Since D is bounded, then, passing to a subsequence (if necessary), we may suppose that $\varphi_j \rightrightarrows \varphi \in \mathcal{O}(\mathbb{D}, \overline{D})$ and $\varphi \neq a$. Using again that D is bounded, we may find an $s \in (0,1)$ such that $\varphi_j(s\mathbb{D}) \subset U$ for any j. It follows by the tautness of $D \cap U$ that $\varphi(s\mathbb{D}) \in \partial D$. Since ∂D contains no affine discs through $a \in \partial D$, we get similarly to the proof of [12, Proposition 7] that $\varphi(s\mathbb{D}) = \{a\}$. Then the identity principle implies that $\varphi = a$ which is a contradiction.

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