NOTE ON COSTARA'S PAPER "ON THE SPECTRAL NEVANLINNA-PICK PROBLEM"

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ABSTRACT. We give a new proof of characterization of the symmetrized polydisc using the notion of polar derivative.

1. Introduction

Let $s_l, l \geq 1$ be the *l-th elementary symmetric function*, that is $s_l(z) = s_l(z_1, \dots, z_n) = \sum_{1 \leq k_1 < \dots < k_l \leq n} z_{k_1} \cdots z_{k_l}$. For $n \geq 1$, let $s : \mathbb{C}^n \to \mathbb{C}^n$ be the function of symmetrization given by the formula

$$s(z_1, \dots, z_n) = \big(s_1(z_1, \dots, z_n), \dots, s_n(z_1, \dots, z_n)\big).$$

Recall that the map $s|_{\mathbb{D}^n} : \mathbb{D}^n \to s(\mathbb{D}^n) =: \mathbb{G}_n$ is a proper holomorphic one (see e.g. [7]), and its image $\mathbb{G}_n$ is called *symmetrized polydisc*. In 2004 Costara gave some characterizations of the symmetrized polidyscs (see [3]). With any point of the symmetrized polydisc he associate some rational functions: one over $\overline{\mathbb{D}}$ and another over $\overline{\mathbb{D}}^{n-1}$, which are closely connected with its geometry. However, in [3] we can not find the way how they arised. Our aim is to enclose us to that aim, and extend a little the results from [3]. Polar derivative turns out to be helphful tool for this purpose.

The symmetrized polydisc appeared in the theory of μ-syntesis (see e.g. [1]) and turned out to be an important object in the geometric function theory (see e.g. [4]). The symmetrized bidisc because of its interesting properties was intensively investigated by many authors, especially by Agler, Costara, Jarnicki, Pflug, Young, Zwonek (some of the papers are listed below). It seems to play an important role not only in complex analysis (it is the first known example of non-convex

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domain for whose the Lempert theorem holds) but also in solving Pick-Nevanlinna Interpolation Problem for $n = 2$ (see e.g. [2]).

2. Definition and basic properties of polar derivative

By a *circular domain* we mean closed interior or exterior of any disc or halfplane and by *circle* boundary of any circular domain.

Let $z_1, \ldots, z_n$ be arbitrary complex numbers (not necessarily finite), $z \neq z_j$, $j = 1, \ldots, n$ and let $m_1, \ldots, m_n$ be non-negative numbers (masses) of total sum (mass) 1 which are placed at points $z_1, \ldots, z_n$, respectively. Choose any linear fractional transformation of complex plane L which sends z to ∞ (that is L is of the form $\frac{az+b}{cz+b}$). *By center of gravity* ζ of such a mass-distribution with respect to z we understand a point $\zeta := \zeta_z$, which is unique, if $L(\zeta)$ is an ordinary center of gravity of $L(z_1), \ldots, L(z_n)$ with masses $m_1, \ldots, m_n$. Note that point ζ does not depend on the choice of L. It is worth mentioning that ordinary center of gravity is a case when $z = \infty$.

Consider all possible mass disstributions with total mass 1 over the fixed points $z_1, \ldots, z_n$ and the point of reference z distinct from all z_ν. Set C_z consisting of the centers of gravity ζ_z of all mass distributions of this kind is called a *circular-arc polygon*. Geometrical interpretation of that definition is contained in

Lemma 1 ([6])**.** *For any two points $w_1, w_2 \in C_z$ arc of circle through w_1, w_2, z with end-points w_1, w_2 that does not contain z, is contained in C_z.*

A set which with the property described in Lemma 1 is called *circularly-convex with respect to z*. The set C_z is the smallest circularly-convex domain with respect to z that contains the points $z_1, \ldots, z_n$. When $z = \infty$, C_z is just a convex hull $\operatorname{conv}(z_1, \ldots, z_n)$, and circular-convexity is reduced to convexity in an ordinary sense.

Note that every circular domain C is circularly-arc convex with respect to any point outside or on C. So, we get

Lemma 2. *If the points $z_1, \ldots, z_n$ lie in a circular domain C but z lies in the complement circular domain to C, then $C_z \subset C$.*

From now on, by the center of gravity we mean the center with special mass distribution $m_1 = \ldots = m_n = \frac{1}{n}$.

Lemma 3 ([6])**.** *Let ζ_z be a center of gravity of $z_1, \ldots, z_n$ with respect to z. Every circle through z and ζ_z separates the points $z_1, \ldots, z_n$ or all the points lie on the circle. Moreover, if $z_1, \ldots, z_n$ belong to a circular domain C, then points z, ζ_z cannot both lie outside C, exception case $z_1 = \ldots = z_n = z = \zeta_z$.*

Let f be any polynomial of degree n:

$$f(z) = C(n,0)A_0^{(0)} + C(n,1)A_1^{(0)}z + \ldots + C(n,n)A_n^{(0)}z^n, \tag{1}$$

where $C(n,k)$ is the binomial coefficient (it is possible that $A_n^{(0)} = \ldots = A_{n-k+1}^{(0)} = 0$, $A_{n-k}^{(0)} \neq 0$, and then ∞ is interpreted as a k-fold zero of f). The point ζ_z is defined as a *center of gravity of a polynomial with respect to* z, if it is a center of gravity of its zeros with respect to z.

Take any point ζ. *Polar derivative* of f with respect to z is

$$(\zeta - z)f'(z) + nf(z) =: A_\zeta f(z) \text{ if } \zeta \text{ is finite,}$$

or just $f'(z)$ if $\zeta = \infty$. Notice that $\deg A_\zeta f < \deg f$ if $A_n^{(0)} \neq 0$. Let points $\zeta_1, \ldots, \zeta_{k+1}$ be given, $(k+1)-$*th polar derivative* f is defined as:

$$A_{\zeta_1} \ldots A_{\zeta_{k+1}} f := A_{\zeta_{k+1}}(A_{\zeta_1} \ldots A_{\zeta_k} f).$$

In fact, the order of points $\zeta_1, \ldots, \zeta_{k+1}$ is not important, that is the operations A_{ζ_1} and A_{ζ_2} are commutative. Actually using induction one might show

$$A_{\zeta_1} \ldots A_{\zeta_k} f(z) = C(n,k)k! \sum_{j=0}^{n-k} C(n-k,j) A_j^{(k)} z^j, \tag{2}$$

where

$$A_j^{(k)} = \sum_{l=0}^{k} \widetilde{s}_l^{(k)}(\zeta_1, \ldots, \zeta_k) A_{j+l}^{(0)} \tag{3}$$

and if points $\zeta_1 = \ldots = \zeta_m = \infty$ and only this then $\widetilde{s}_l^{(k)}(\zeta_1, \ldots, \zeta_k) := 0$ for $l < m$ and $\widetilde{s}_l^{(k)}(\zeta_1, \ldots, \zeta_k) := s_{l-m}^{(k-m)}(\zeta_{m+1}, \ldots, \zeta_k)$, and the last one are elementary symmetric polynomials.

For derivative f' of a polynomial f there is a well known Gauss-Lucas theorem, which says that every convex set which contains all zeros of f, also contains its critical points. For polar derivative similar results holds, which is in fact contained in Lemma 3, and which implies Gauss-Lucas theorem. Namely

Theorem 1 (Laguerre)**.** *If all the zeros of the n-th degree polynomial $f(z)$ lie in a circular domain C and if Z is any zero of $A_\zeta f$, then not both points Z, ζ may lie outside C. Furthermore, if $f(Z) \neq 0$, then any circle through Z and ζ either passes through all the zeros of f or separates these zeros.*

We say that polynomial g is *apolar* to polynomial f (both of them are of degree n) if nth polar derivative of f counted with respect to the zeros of the $g(z)$ vanishes. Notice that g is apolar to f if and only if f is apolar to g, and we express this fact saying that f and g are apolar.

Lemma 4. [6, pg. 60] *Let f (as in (1)) be apolar to g, where*

$$g(z) = \sum_{j=0}^{n} C(n,j) B_j^{(0)} z^j,$$

then every two circularly-arc polygons that are circularly convex with respect to the same point and that contain all the zeros of $f(z)$ and $g(z)$ respectively, also have at least one common point.

3. Statements and proofs

Let $(s_1, \ldots, s_n) \,\epsilon\, \mathbb{C}^n$. Define $P(z) = z^n - s_1 z^{n-1} + \ldots + (-1)^n s_n$. The [3, Theorem 3.1] can be generalised as follows

Proposition 1. $P^{-1}(0) \subseteq \mathbb{D}(z_0, r)$ *if and only if*

$$\sup_{z:|z-z_0|\geq r} \left| \frac{A_{z_0}P(z)}{P'(z)} \right| =: f(z) < r. \tag{4}$$

Let $P(z) = \sum_{k=0}^{n} C(n,k) a_j z^j$, then

$$\frac{A_{z_0}P(z)}{P'(z)} = \frac{\sum_{k=0}^{n-1} C(n-1,k) a_k z^k}{\sum_{k=0}^{n-1} C(n-1,k) a_{k+1} z^k}. \tag{5}$$

In [3] it is the case when $z_0 = 0$ and $r = 1.$,

Proof. Let points $z_1, \ldots, z_n$ be all zeros of $P(z)$ and fix any z outside or on C. Then, in view of Lemma 3, it is enough to notice that

$$\left| \frac{A_{z_0}P(z)}{P'(z)} \right| = |\zeta_z - z_0| \leq \operatorname{dist}(P^{-1}(0), \partial\mathbb{D}(z_0, r)) < r, \tag{6}$$

where ζ_z is a center of gravity $P(z)$ with respect to z. □

Using the same argument as above we get:

Corollary 1. $P^{-1}(0) \subseteq \overline{\mathbb{D}}(z_0, r)$ *if and only if*

$$\sup_{z:|z-z_0|\geq r} \left| \frac{A_{z_0}P(z)}{P'(z)} \right| \leq r.$$

Corollary 2. $P^{-1}(0) \subseteq \partial\mathbb{D}(z_0, r)$ *if and only if* $\zeta_z \,\epsilon\, \partial\,\mathbb{D}(z_0, r)$ *and* $(P')^{-1}(0) \subseteq \overline{\mathbb{D}}(z_0, r)$ *for all* $z \,\epsilon\, \partial\mathbb{D}(z_0, r)$.

Proof. Assume that points $P^{-1}(0)$ lie on a circle $|z - z_0| = r$, so ζ_z also lies on this circle. Conversly, from Corollary 1 we obtain $P^{-1}(0) \subseteq \overline{\mathbb{D}}(z_0, r)$. If $P(\widetilde{z}) = 0$, then $\widetilde{z}$ must lies on the boundary of that disc. Indeed, otherwise $|\zeta_z - z_0| < r$ for any $z \,\epsilon\, \partial\overline{\mathbb{D}} \setminus P^{-1}(0)$. □

Similar condition could be writen for those $P(z)$ whose zeros are in $\overline{\mathbb{D}}(z_0, r)$ but neither in nor on $|z - z_0| = r$.

It was characterization of $\mathbb{G}_n$ over unit disc. To get characterization over $\mathbb{G}_{n-1}$ we use $n-1$th polar derivative.

Proposition 2. *$P^{-1}(0) \subseteq \mathbb{D}(z_0, r)$ if and only if there exists $0 < s < r$ such that the only zero of $A_{\zeta_1} \ldots A_{\zeta_{n-1}} P$ is in $\mathbb{D}(z_0, s)$ for all $\zeta_1, \ldots, \zeta_{n-1} \epsilon \notin \mathbb{D}(z_0, r)$.*

Lemma 6 is an anologue of Theorem 3.5 in [3].

Proof. The only zero of $A_{\zeta_1} \ldots A_{\zeta_{n-1}} P$ is

$$-\frac{A_{z_0} A_{\zeta_1} \ldots A_{\zeta_{n-1}} P}{A_\infty A_{\zeta_1} \ldots A_{\zeta_{n-1}} P} =: g(\zeta_1, \ldots, \zeta_{n-1}).$$

Of course, $g(\zeta, \ldots, \zeta) = f(\zeta)$ where f is as in Lemma 5. Applying (k-1)-times Lemma 3 gives 'only if'. It remains to show the sufficiency of the above condition. For this part, notice that (2) and (3) imply $A_{\widetilde{z}}^{n-1} P(\widetilde{z}) = P(\widetilde{z})$ for any $\widetilde{z}$. $\square$

Lemma 3 gives the following generalization of Proposition 1 and 2 which is the main result in this paper. It extends the main result in [3].

Proposition 3. *Let f be any polynomial of degree n with coefficient at z^n equal 1. The following assertions are equivalent:*

(1) $P^{-1}(0) \subset \mathbb{D}(z_0, r)$;

(2)
$$\sup_{z \notin \mathbb{D}(z_0, r)} \left| \frac{A_{z_0} A_{\zeta_1} \ldots A_{\zeta_{k-1}} f(z)}{A_\infty A_{\zeta_1} \ldots A_{\zeta_{k-1}} f(z)} \right| < r$$
for any positive integer number $1 \leqslant k \leqslant n-1$ and any choice of the points $\zeta_1, \ldots, \zeta_{k-1} \notin \mathbb{D}(z_0, r)$;

(3) *(2) holds for $k = 1$;*

(4) *(2) holds for $k = n-1$;*

(5) *(2) holds for $k = n-1$ and $\zeta_1 = \ldots = \zeta_{n-1} \notin \mathbb{D}(z_0, r)$;*

(6) *(2) holds for some $1 \leqslant k \leqslant n-1$;*

(7) *(2) holds for some $1 \leqslant k \leqslant n-1$, $\zeta_1 = \ldots = \zeta_k \notin \mathbb{D}(z_0, r)$.*

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