BERGMAN COMPLETENESS IS NOT A QUASI-CONFORMAL INVARIANT

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Abstract. We show that Bergman completeness is not a quasi-conformal invariant for general Riemann surfaces.

Recently, B.-Y. Chen (see [Chen 1]) asked a question whether Bergman completeness is a quasi-conformal invariant of Riemann surfaces. In this paper we will give an example showing that the answer to the question is negative.

The original idea of the example is based on the paper [Zwo 1]. In [Zwo 1], the growth of the Bergman kernel near the boundary has been estimated with the help of potential-theoretical quantities. Also in [Ju], a necessary and sufficient condition for Bergman completeness of Zalcman type domains has been found. The present paper will use the methods in [Pfl-Zwo].

For $0 < r < \frac{1}{4}$ and $t \in \left(0, \frac{1}{2}\right)$ define

$$A^{r,t} := \bigcup_{k=1}^{\infty} A_k^{r,t} \cup \{0\}$$

where

$$A_k^{r,t} := \{r^k e^{i\theta} : -2\alpha_k \leq \theta \leq 2\alpha_k\}$$

and $\sin \alpha_k = e^{-t^{-k}}$.

Finally we put $D^{r,t} := \triangle(0,1) \setminus A^{r,t}$ where $\triangle(p,r) := \{z \in \mathbb{C} : |z-p| < r\}$, $p \in \mathbb{C}$, $r > 0$.

Theorem 1.

(1) $D^{r,t}$ is Bergman exhaustive if and only if $r^2 \leq t$;
(2) if $t \leq r^4$ then $D^{r,t}$ is not Bergman complete.

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Corollary 2. *Bergman completeness is not in general a quasi-conformal invariant for Riemann surfaces.*

Recall that a homeomorphism f defined on a domain in the complex plane is called L $(L > 1)$ quasi-conformal if it is differentiable almost everywhere and

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq \frac{L-1}{L+1}\left|\frac{\partial f}{\partial z}\right|.$$

Before we present the proofs of Theorem 1 and Corollary 2 let us recall some notions and results which we need in the paper.

In order to prove Theorem 1, we need some lemmas on logarithmic capacity and Bergman kernel (see [Jar-Pfl]). First we recall necessary notions from potential theory (see [Ran] where also other properties of potential-theoretic objects that we use are given).

For a probabilistic measure μ defined on all Borel subsets of a compact set K (denote $\mu \in \mathcal{P}(K)$), we define its logarithmic potential by

$$p_\mu(z) := \int_K \log|z-w| d\mu(w), \ z \in \mathbb{C}.$$

Recall that p_μ is harmonic in $\mathbb{C} \setminus K$ and subharmonic in $\mathbb{C}$.

We also denote the energy of μ as follows

$$I(\mu) := \int_K p_\mu(z) d\mu(z) = \int_K \int_K \log|w-z| d\mu(w) d\mu(z).$$

A probabilistic measure ν defined on Borel subsets of the compact set K is called *the equilibrium measure of* K if $I(\nu) = \sup\{I(\mu) : \mu \in \mathcal{P}(K)\}$. It is well-known that the equilibrium measure exists and is unique if K is not polar (a set $F \subset \mathbb{C}$ is called polar if there is a subharmonic function u defined on $\mathbb{C}$ such that $u \not\equiv -\infty$ and $F \subset \{u = -\infty\}$).

The logarithmic capacity of a subset E of $\mathbb{C}$ is given by the formula

$$\operatorname{cap}(E) := e^{\sup\{I(\mu):\mu\in\mathcal{P}(K), K \text{is a compact subset of } E\}}.$$

In case when K is compact and not polar then $\operatorname{cap}(K) = e^{I(\nu)}$, where ν denotes the equilibrium measure of K. It is well-known that a Borel set $E \subset \mathbb{C}$ is polar iff $\operatorname{cap}(E) = 0$.

Now let us recall the notion of the Bergman kernel. Let D be a bounded domain in $\mathbb{C}^n$. Denote by $L^2_h(D)$ square integrable holomorphic functions on D. $L^2_h(D)$ is a Hilbert space with the scalar product induced from $L^2(D)$ denoted by $\langle \cdot, \cdot \rangle_D$ (we also denote by $||\cdot||_D$ the L^2 norm on D and the space of square integrable holomorphic functions on D is denoted by $L^2_h(D)$).

Let us define the *Bergman kernel of* D as

$$K_D(z) = \sup\{\frac{|f(z)|^2}{||f||^2_D} : f \in L^2_h(D), f \not\equiv 0\}, z \in D$$

and the fundamental form of the Bergman metric as

$$B_D(z) = i\partial\overline{\partial}\log K_D(z), z \in D$$

Let

$$I_D(z,X) = \sup\{\frac{|f'(z)X|^2}{||f||_D^2} : f \in L_h^2(D), f(z) = 0, f \not\equiv 0\}, z \in D, X \in \mathbb{C}^n.$$

The following result is classical

$$B_D(z)(X) = \frac{I_D(z,X)}{K_D(z)}, z \in D, X \in \mathbb{C}^n.$$

Let D be a bounded domain in $\mathbb{C}^n$, $z_0 \in \partial D$. Then D is *Bergman exhaustive at* $z_0 \in \partial D$ if $\lim_{D \ni z \to z_0} K_D(z) = \infty$. We call D *Bergman exhaustive* if D is Bergman exhaustive at z_0 for any $z_0 \in \partial D$.

A bounded domain D is called *Bergman complete* if any Cauchy sequence with respect to the Bergman distance is convergent to some point in D under the standard topology of D. For references on the Bergman kernel, metric and distance, see [Chen 2].

It is known that if a bounded domain $D \subset \mathbb{C}$ is Bergman exhaustive then D is Bergman complete (see [Chen 3]), the converse implication does not hold in general (see [Zwo 2]). Also bounded hyperconvex domain is Bergman exhaustive (see [Ohs]) and Bergman complete (see [B-P], [Her]).

Now let us introduce the notions necessary for the description of Bergman exhaustive points in dimension one.

For a bounded domain $D \subset \mathbb{C}$ and for a point $z \in \bar{D}$, we introduce the following potential theoretic quantity;

$$\gamma_D^{(n)}(z) := \int_0^{1/4} \frac{d\delta}{\delta^{2n+3}(-\log(\mathrm{cap}(\bar{\triangle}(z,\delta) \setminus D))}.$$

The following lemma comes from the paper [Zwo 1].

Lemma 3. *Let D be a bounded domain in $\mathbb{C}$ and let $z_0 \in \partial D$. Then*

$$\lim_{D \ni z \to z_0} \gamma_D^{(0)}(z) = \infty \tag{1}$$

if and only if

$$D \text{ is Bergman exhaustive at } z_0. \tag{2}$$

Remark 4. It also follows from classical results that the domain $D^{r,t}$ is Bergman exhaustive at all of its boundary point except for 0.

Since we shall only consider bounded domains in $\mathbb{C}$, for simplicity we denote $\beta_D(z) = B_D(z)(1), z \in D$. From the paper [Pfl-Zwo], we know

Lemma 5. *Let D be a bounded domain in $\mathbb{C}$, $D \ni z_k \to z_0 \in \partial D$. If*

$$\limsup_{k\to\infty} \gamma_D^{(1)}(z_k) < \infty,$$

then

$$\limsup_{k\to\infty} \beta_D(z_k) < \infty.$$

Now we move to the proofs of main results. Let us first see how we derive Corollary 2 from Theorem 1. Then we show Theorem 1.

Proof of Corollary 2. For $\alpha > \frac{1}{2}$ we define

$$\varphi(z) = z^{\alpha}\overline{z}^{\alpha-1}.$$

Note that φ is a quasi-conformal mapping from $D^{r,t}$ to $D^{r^{2\alpha-1},t}$. Choosing for instance $r = 1/8$, $\alpha = 2/3$, $t = 1/32$ we get that Bergman completeness is not a quasi-conformal invariant. Actually $D^{r,t}$ is, in view of Theorem 1 (1), Bergman exhaustive and thus it is Bergman complete whereas $D^{r^{2\alpha-1},t}$ is, in view of Theorem 1 (2), not Bergman complete. $\square$

Remark 6. By Theorem 1 we also can deduce that Bergman exhaustiveness is not a quasi-conformal invariant. But we still do not know whether it is a conformal invariant for bounded domains in the complex plane. Also due to the above theorem, there are lots of domains which are Bergman complete but not Bergman exhaustive (see [Zwo 2]).

Note also that the example from Corollary 2 is a Riemann surface with infinite dimensional fundamental group. But B.-Y. Chen informed the author that Bergman completeness is a quasi-conformal invariant when the fundamental group is finitely generated.

Behind Chen's question on the quasi-conformal invariance of the Bergman completeness, there is an affirmative result by Pfluger on the quasi-conformal invariance of the existence of the Green function, and a very deep negative result by Beurling-Ahlfors on the invariance of the nullity of linear measure for subsets of the circle as the boundary of the disc (see [Sa-Nak]). The author would like to thank the referee for this comment.

Proof of Theorem 1.

If $D^{r,t}$ is Bergman exhaustive then by Lemma 3,

$$\lim_{D^{r,t}\ni z\to 0} \gamma^{(0)}_{D^{r,t}}(z) = \infty.$$

For any $\delta \in [0, 1/4]$, if $-1 < x < 0$ and $|x|$ small enough we have

$$\bar{\triangle}(x,\delta) \setminus D^{r,t} \subset \bar{\triangle}(0,\delta) \setminus D^{r,t}.$$

Thus for such x

$$\gamma^{(n)}_{D^{r,t}}(x) \leq \gamma^{(n)}_{D^{r,t}}(0),$$

consequently, we get $\gamma^{(0)}_{D^{r,t}}(0)=\infty$. Now choose

$$A_1=\{z\in\mathbb{C};2r^2\le |z|\le 1/4\},$$

$$A_k=\{z\in\mathbb{C};2r^{k+1}\le |z|\le 2r^k\}, k\ge 2.$$

We have

$$\gamma^{(n)}_{D^{r,t}}(0)=(\int_{2r^2}^{1/4}+\sum_{k=2}^{\infty}\int_{2r^{k+1}}^{2r^k})\frac{d\delta}{\delta^{2n+3}(-\log(\mathrm{cap}(\bar{\triangle}(0,\delta)\setminus D^{r,t}))}.$$

Let

$$C_1=\int_{2r^2}^{1/4}\frac{d\delta}{\delta^{2n+3}(-\log(\mathrm{cap}(\bar{\triangle}(0,\delta)\setminus D^{r,t}))},$$

$$C_k=\int_{2r^{k+1}}^{2r^k}\frac{d\delta}{\delta^{2n+3}(-\log(\mathrm{cap}(\bar{\triangle}(0,\delta)\setminus D^{r,t}))}, k\ge 2.$$

Then we have

$$(1/4-2r^2)4^{2n+3}\frac{1}{-\log(\mathrm{cap}(A_2\setminus D^{r,t}))}\le C_1,$$

$$C_1\le (1/4-2r^2)(2r^2)^{-2n-3}\sum_{j=1}^{\infty}\frac{1}{-\log(\mathrm{cap}(A_j\setminus D^{r,t}))},$$

$$(2r^k)^{-2n-2}(1-r)\frac{1}{-\log(\mathrm{cap}(A_{k+1}\setminus D^{r,t}))}\le C_k,$$

$$C_k\le (2r^{k+1})^{-2n-2}(1/r-1)\sum_{j=k}^{\infty}\frac{1}{-\log(\mathrm{cap}(A_j\setminus D^{r,t}))}, k\ge 2.$$

As $\mathrm{cap}(A_j\setminus D^{r,t})=r^je^{-t^{-j}}$, we get

$$\frac{t^j}{1-t\log r}\le\frac{1}{-\log(\mathrm{cap}(A_j\setminus D^{r,t}))}\le t^j.$$

Thus there is a sufficiently large constant $C(t,r,n)$ such that

$$C(t,r,n)^{-1}(\frac{t}{r^{2n+2}})^k\le C_k\le C(t,r,n)(\frac{t}{r^{2n+2}})^k.$$

Now due to the lower semi-continuity of $\gamma^{(0)}_{D^r}$ ([Zwo 1]), the theorem follows easily from Lemma 3 (part (1)) and Lemma 6 (part (2)). □

References

[B-P] Z. Błocki & P. Pflug, *Hyperconvexity and Bergman completeness*, Nagoya Math. J. **151,** (1998), 221–225.

[Chen 1] B.-Y. Chen, *An essay on Bergman completeness, preprint* (2011).

[Chen 2] B.-Y. Chen, *A remark on the Bergman completeness*, Complex Variables Theory Appl. **42, no. 1** (2000), 11–15.

[Chen 3] B.-Y. Chen, *Completeness of the Bergman kernel on non-smooth pseudoconvex domains*, Ann. Polon. Math. **LXXI(3)** (1999), 242–251.

[Her] G. Herbort, *The Bergman metric on hyperconvex domains*, Math. Z. **232(1)** (1999), 183–196.

[Jar-Pfl] M. Jarnicki & P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, Berlin, 1993.

[Ju] P. Jucha, *Bergman completeness of Zalcman type domains*, Studia Math. **163** (2004), 71–83.

[Ohs] T. Ohsawa, *On the Bergman kernel of hyperconvex domains*, Nagoya Math. J. **129** (1993), 43–52.

[Pfl-Zwo] P. Pflug & W. Zwonek, *Logarithmic capacity and Bergman functions*, Arch. Math. (Basel) **80** (2003), 536–552.

[Ran] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, 1995.

[Sa-Nak] L. Sario & M. Nakai, *Classification Theory of Riemann Surfaces*, Springer, 1970.

[Zwo 1] W. Zwonek, *Wiener's type criterion for Bergman exhaustiveness*, Bull. Polish Acad. Sci. Math. **50** (2002), 297–311.

[Zwo 2] W. Zwonek, *An example concerning Bergman completeness*, Nagoya Math. J. **164** (2001), 89–102.

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