

Spectral monodromy in classical completely integrable or quasi-integrable cases

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Abstract

The spectral monodromy is a combinatorial invariant, defined directly from the spectrum of certain non-selfadjoint classical operators. It is an obstruction against the global lattice structure of the spectrum, seen as a discrete subset of points in the complex plane, and in the semi-classical limit. We work with small non-selfadjoint perturbations of classical selfadjoint operators with two degrees of freedom, assuming that the (semi-)classical principal symbol of the unperturbed part is in two different cases: completely integrable system in the first case, and quasi-integrable one with a globally (non-degenerate) isoenergetic condition in the second case. In each case, the spectral monodromy allows to recover the corresponding classical monodromy.

Keywords: Hamiltonian systems, monodromy, non-selfadjoint, asymptotic spectral, pseudo-differential operators, KAM theory.

Contents

1	Introduction and motivation	2
2	Spectral monodromy in the completely integrable case	5
2.1	Spectral asymptotic for small perturbed non-selfadjoint operators	6
2.1.1	General assumptions	6
2.1.2	Asymptotic expansion of eigenvalues	10
2.2	Spectral asymptotic pseudo-lattice and its monodromy	13
2.3	Spectral monodromy recovers the classical monodromy	17
3	Spectral monodromy in the quasi-integrable case	19
3.1	Quasi-periodic flows of quasi-integrable systems	19
3.2	Spectral monodromy of $P_{\varepsilon,\lambda}$	21
3.3	Discussions	25

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1 Introduction and motivation

We are interested in the global structure of the spectrum of non-selfadjoint h -Weyl-pseudodifferential operators with two degrees of freedom, in the semi-classical limit, that is, when the small classical parameter h tends to 0. We will define in this paper a feature—the *spectral monodromy* associated to the spectrum, seen as a discrete subset of points in the complex plane. The considered operators are small non-selfadjoint perturbations of selfadjoint operators, with certain different assumptions on the classical dynamic of the unperturbed part.

The spectral monodromy was defined the first time in [35] for classes of classical operators, associated with completely integrable systems. In that work, one assumed that, the principal symbol of the unperturbed part and the leading term of the perturbed part are in involution for the Poisson bracket. This is a particular case of the present work which is more interesting and quite developed. We assume in this paper that the classical flow of the unperturbed part is in two cases: completely integrable in the first case, and quasi-integrable with a globally non-degenerate condition in the second case.

In the paper [10], the authors succeeded to define a monodromy related to quasi-integrable systems. They proposed a mysterious question: whether a quantum invariant (like monodromy) can be defined for classical operators in the quasi-integrable case. This is also a motivation of our work. The results obtained in the second case of this paper answer completely that question.

We give now a brief description for the monodromy spectral. The theory of spectral asymptotics (for example Refs. [5], [6], [7], and especially [8]) showed that under certain technical general hypothesis, the spectrum of such operators in domains depending on the classical parameter of the complex spectral plane has *microlocally* a deformed lattice structure. That means it is image of a square lattice by a sort of local chart, called a *micro chart*. Such lattices are of small size but they can be defined around many points in the spectral domain. A family of close micro-lattices in a small domain forms a local lattice, on which we have a corresponding family of micro-charts, called a *pseudo-chart*. The pseudo-charts play the role of local charts on the spectral domain.

A natural question we propose is that: whether the spectrum has globally a lattice structure? The spectral monodromy which is a combinatorial invariant related to the spectrum, allows us to answer this question. It is given when we analyze how the local spectral lattices are glued, by examining transition maps between overlapping pseudo-charts. In fact, these local lattices are glued them-self by a special structure, which is characterized by the spectral monodromy. It is defined as a element of the Čech cohomology group $\check{H}^1(U, GL(2, \mathbb{Z}))$, independent of classical parameter and given small perturbations. When the spectral monodromy is not trivial, then the transition maps either, and the spectrum hasn't a smooth global lattice structure.

There exists a known quantum invariant, that is the quantum monodromy given by Vu Ngoc, see [26]. It is defined for the joint spectrum of integrable quantum systems of $n \geq 2$ commuting selfadjoint classical operators. Contrariwise, the spectral monodromy

is defined for a single non-selfadjoint operator. However the quantum monodromy stills useful for our work. It give simple examples of the existence of monodromy for the spectrum. In fact, when a non-selfadjoint operator is normal, then its discrete spectrum is completely identified with the joint spectrum of an integrable quantum system, which is composed of the real part and the imaginary part of the operator. For more details on this point, we refer to [35] Sec. II. The existence of non-trivial monodromy is known and assured for several integrable quantum systems, for example the quantum spherical pendulum (see [19], and [32]) or the quantum Champagne bottle (see[30], and [31]).

In both cases, we consider Weyl- h -pseudodifferential non-selfadjoint perturbed operators in dimension 2, depending on a small parameter ε such that $h \ll \varepsilon = \mathcal{O}(h^\delta)$, with $0 < \delta < 1$.

In the first case, we treat such a operator of the form $P_\varepsilon = P(x, hD_x, \varepsilon; h)$, where the unperturbed operator $P := P_{\varepsilon=0}$ is formally selfadjoint, assuming that the principal classical symbol p of P is completely integrable. A particular case of this case when the perturbed operator of the form $P_\varepsilon = P + i\varepsilon Q$, assuming that the corresponding principal symbols p of P and q of Q commute for the Poisson bracket, is given in [35] Sec. III.

In the second case, we perturb even the completely integrable symbol p by a small quantity to get a quasi-integrable system. We consider a classical operator $P_{\varepsilon, \lambda}$, depending also smoothly on a small enough parameter $0 < \lambda \ll 1$, and assuming that the principal symbol of the selfadjoint unperturbed operator $P_\lambda := P_{\varepsilon=0, \lambda}$ is a quasi-integrable system of the form $p_\lambda = p + \lambda p_1$, here p_1 is a bounded Hamiltonian. Moreover, a general hypothesis given is the globally isoenergetic non-degeneracy of p .

It knows from the spectral asymptotic theory (see [8]) that, under an ellipticity condition at infinity (see (2.9)), in both cases, the spectrum of the perturbed operators is discrete, and included in a horizontal band of size $\mathcal{O}(\varepsilon)$. Moreover, that work allows give an asymptotic expansion of the spectrum, located in a some small domains of size $\mathcal{O}(h^\delta) \times \mathcal{O}(\varepsilon h^\delta)$ of the spectral band, called good rectangles. These good rectangles are associated with Diophantine invariant Lagrangian tori in the phase space, on which the Hamiltonian flow of the unperturbed part, either p in the first case or p_λ in the second case, is quasi-periodic of constant frequencies.

On the other hand, the existence of such tori is insured and moreover the tori form a set of Cantor type of full measure (in measure theoretic sense) in the phase space. In the first case, it is classical since the fact that the phase space of the completely integrable Hamiltonian system p is foliated by invariant Lagrangian 2-tori (the Liouville tori) and almost of them satisfy a Diophantine condition. And then, for the seconde case, we need the KAM theory which deals quasi-periodic motions in small perturbed Hamiltonian systems (like p_λ). It shows that, under non-degeneracy assumptions and λ small enough, these Diophantine invariant tori for the flow of p persist (slightly deformed) as Diophantine invariant tori (KAM tori) for one of the perturbed system p_λ .

Therefore, we have a lot of good rectangles in the spectral band. The corresponding spectrum μ of the perturbed operators in each good rectangle, denoted by $R^{(a)}(\varepsilon, h)$,

here $a \in \mathbb{R}^2$ is used to fix the good rectangle, is given by

$$R^{(a)}(\varepsilon, h) \ni \mu = P\left(\xi_a + h\left(k - \frac{\eta}{4}\right) - \frac{S}{2\pi}, \varepsilon; h\right) + \mathcal{O}(h^\infty), \quad k \in \mathbb{Z}^2, \quad (1.1)$$

uniformly for small h, ε . Here ξ_a are action coordinates, $S \in \mathbb{R}^2$ is the action integrals, $\eta \in \mathbb{Z}^2$ is the Maslov indices, of the fundamental cycles of the corresponding Diophantine torus. $P(\xi, \varepsilon; h)$ is a smooth function admitting an asymptotic expansion in (ξ, ε, h) .

The spectrum so has the structure of a deformed lattice, with horizontal spacing h and vertical spacing εh . Since (1.1), there is a local diffeomorphism, denoted by f , that sends the spectrum in the rectangle to a part of $h\mathbb{Z}^2$, with modulo $\mathcal{O}(h^\infty)$,

$$R^{(a)}(\varepsilon, h) \ni \mu \mapsto f(\mu; \varepsilon, h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty). \quad (1.2)$$

This map is called a micro-chart. A family (of Cantor type) of close micro-charts on a small domain forms a local pseudo-chart of the spectrum, as discussed at the beginning of this section.

We notice that each micro-chart is normally valid for one good rectangle. However, an important fact we show in this paper that we can build spectral pseudo-charts such that the leading term in asymptotic expansions of their micro-charts in small parameters h, ε is locally well defined. This leading term is well defined for every micro-chart of a pseudo-chart. The construction of micro charts in each case will respectively be done in detail in Sec. 2.2 and Sec. 3.2.

With regard to the global problem, we consider the spectrum as a discrete subset of the complex plane. We can apply a result of [35] about a discrete set, called asymptotic pseudo-lattice (see also Definition 2.15) for the spectrum. It shows that the differential of the transition maps between two overlapping local pseudo-charts $(f_i, U^i(\varepsilon))$ and $(f_j, U^j(\varepsilon))$ is in the group $GL(2, \mathbb{Z})$, with modulo $\mathcal{O}(\varepsilon, \frac{h}{\varepsilon})$:

$$d(\tilde{f}_i) = M_{ij}d(\tilde{f}_j) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}),$$

where $\tilde{f}_i = f_i \circ \chi$, $\tilde{f}_j = f_j \circ \chi$, with χ is the function $(u_1, u_2) \mapsto (u_1, \varepsilon u_2)$, and $M_{ij} \in GL(2, \mathbb{Z})$ is a integer constant matrix.

Let $U(\varepsilon)$ be a bounded open domain in the spectral band and cover it by an arbitrary (small enough) locally finite covering of pseudo-charts $\{(f_j, U^j(\varepsilon))\}_{j \in \mathcal{J}}$, here \mathcal{J} is a finite index set. Then the spectral monodromy on $U(\varepsilon)$ is defined as the unique 1-cocycle $\{M_{ij}\}$, with modulo-coboundary in the first Čech cohomology group. We have just explained the principle idea how to define the monodromy.

This work is a inverse quantum problem. The spectral monodromy is defined directly from the spectrum. However, as an another important result, it is strictly related to the classical problem, and the classical results, in turn, illuminate again the initial quantum problem. In both cases, the spectral monodromy allows us to recover known classical

invariants. In the first case, the spectral monodromy can be identified to the classical monodromy for the Liouville invariant tori of integrable systems, given in [16]. And in the second case, the spectral monodromy allows to recover the monodromy of the KAM invariant tori of quasi-integrable systems, defined by Broer and co-workers [10]. Then the geometry of corresponding principal symbols plays an important role in quantum properties of classical operators.

2 Spectral monodromy in the completely integrable case

For the first case, in this section we shall define the spectral monodromy, associated with the spectrum of small non-selfadjoint perturbations of selfadjoint classical operators in two dimensions, assuming that the classical flow of the unperturbed part is completely integrable.

We will work throughout this article with pseudodifferential operators obtained by the h -Weyl-quantization of a standard space of symbols on $T^*M = \mathbb{R}_{(x,\xi)}^{2n}$, here $M = \mathbb{R}^n$ or a compact manifold of n dimensions, and in particular $n = 2$. We denote σ the standard symplectic 2-form on T^*M .

In the following, we introduce classical operators on $M = \mathbb{R}^n$, but it is alright general to the manifold case.

Definition 2.1. *A function $m : \mathbb{R}^{2n} \rightarrow (0, +\infty)$ is called an order function if there are constants $C, N > 0$ such that*

$$m(X) \leq C \langle X - Y \rangle^N m(Y), \forall X, Y \in \mathbb{R}^{2n},$$

with notation $\langle Z \rangle = (1 + |Z|^2)^{1/2}$ for $Z \in \mathbb{R}^{2n}$.

Definition 2.2. *Let m be an order function and $k \in \mathbb{R}$, we define classes of symbols of h -order k , $S^k(m)$ (families of functions) of $(a(\cdot; h))_{h \in (0,1]}$ on $\mathbb{R}_{(x,\xi)}^{2n}$ by*

$$S^k(m) = \{a \in C^\infty(\mathbb{R}^{2n}) \mid \forall \alpha \in \mathbb{N}^{2n}, \quad |\partial^\alpha a| \leq C_\alpha h^k m\}, \quad (2.3)$$

for some constant $C_\alpha > 0$, uniformly in $h \in (0, 1]$.

A symbol is called $\mathcal{O}(h^\infty)$ if it's in $\cap_{k \in \mathbb{R}} S^k(m) := S^\infty(m)$.

Then $\Psi^k(m)(M)$ denotes the set of all (in general unbounded) linear operators A_h on $L^2(\mathbb{R}^n)$, obtained from the h -Weyl-quantization of symbols $a(\cdot; h) \in S^k(m)$ by the integral:

$$(A_h u)(x) = (Op_h^w(a)u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\xi} a\left(\frac{x+y}{2}, \xi; h\right) u(y) dy d\xi. \quad (2.4)$$

We refer to Refs. [22], [23], and [24] for the theory of classical operators. In this paper, we always assume that symbols admit a classical asymptotic expansion in integer powers of h . The leading term in this expansion is called the principal symbol of operators.

2.1 Spectral asymptotic for small perturbed non-selfadjoint operators

In this section, we recall some results of the spectral asymptotic theory for small non-selfadjoint perturbations of selfadjoint classical operators in two dimensions.

2.1.1 General assumptions

We give here general assumptions of small non-selfadjoint perturbations of selfadjoint operators in two dimensions, as in Refs. [8], [5], and [6].

M denotes \mathbb{R}^2 or a connected compact analytic real (Riemannian) manifold of dimension 2 and we denote by \widetilde{M} the canonical complexification of M , which is either \mathbb{C}^2 in the Euclidean case or a Grauert tube in the case of manifold (see Ref. [33]).

We consider a non-selfadjoint h -pseudodifferential operator P_ε on M and suppose that

$$P_{\varepsilon=0} := P \quad \text{is formally selfadjoint.} \quad (2.5)$$

Note that if $M = \mathbb{R}^2$, the volume form $\mu(dx)$ is naturally induced by the Lebesgue measure on \mathbb{R}^2 . If M is a compact Riemannian manifold, then the volume form $\mu(dx)$ is induced by the given Riemannian structure of M . Therefore in both cases the volume form is well defined and the operator P_ε may be seen as an (unbounded) operator on $L^2(M, \mu(dx))$. We always denote the principal symbol of P_ε by p_ε which is defined on T^*M .

We will assume the ellipticity condition at infinity for P_ε at some energy level $E \in \mathbb{R}$ as follows:

When $M = \mathbb{R}^2$, let

$$P_\varepsilon = P(x, hD_x, \varepsilon; h) \quad (2.6)$$

be the Weyl quantification of a total symbol $P(x, \xi, \varepsilon; h)$ depending smoothly on ε in a neighborhood of $(0, \mathbb{R})$ and taking values in the space of holomorphic functions of (x, ξ) in a tubular neighborhood of \mathbb{R}^4 in \mathbb{C}^4 on which we assume that:

$$|P(x, \xi, \varepsilon; h)| \leq \mathcal{O}(1)m(\operatorname{Re}(x, \xi)). \quad (2.7)$$

Here m is an order function in the sense of Definition 2.1. We assume moreover that $m > 1$ and P_ε is classical of order 0,

$$P(x, \xi, \varepsilon; h) \sim \sum_{j=0}^{\infty} p_{j,\varepsilon}(x, \xi)h^j, \quad h \rightarrow 0, \quad (2.8)$$

in the selected space of symbols. In this case, the principal symbol $p_\varepsilon = p_{0,\varepsilon}$ is the first term of the above expansion, and the ellipticity condition at infinity is

$$|p_{0,\varepsilon}(x, \xi) - E| \geq \frac{1}{C}m(\operatorname{Re}(x, \xi)), \quad |(x, \xi)| \geq C, \quad (2.9)$$

for some $C > 0$ large enough.

When M is a compact manifold, we consider P_ε a differential operator on M such that in local coordinates x of M , it is of the form:

$$P_\varepsilon = \sum_{|\alpha| \leq m} a_{\alpha, \varepsilon}(x; h)(hD_x)^\alpha, \quad (2.10)$$

where $D_x = \frac{1}{i} \frac{\partial}{\partial x}$ and $a_{\alpha, \varepsilon}$ are smooth functions of ε in a neighborhood of 0 with values in the space of holomorphic functions on a complex neighborhood of $x = 0$.

We assume that these $a_{\alpha, \varepsilon}$ are classical of order 0,

$$a_{\alpha, \varepsilon}(x; h) \sim \sum_{j=0}^{\infty} a_{\alpha, \varepsilon, j}(x) h^j, h \rightarrow 0, \quad (2.11)$$

in the selected space of symbols. In this case, the principal symbol p_ε in the local canonical coordinates associated (x, ξ) on T^*M is

$$p_\varepsilon(x, \xi) = \sum_{|\alpha| \leq m} a_{\alpha, \varepsilon, 0}(x) \xi^\alpha, \quad (2.12)$$

and the ellipticity condition at infinity is

$$|p_\varepsilon(x, \xi) - E| \geq \frac{1}{C} \langle \xi \rangle^m, (x, \xi) \in T^*M, |\xi| \geq C, \quad (2.13)$$

for some $C > 0$ large enough. Notice here that M has a Riemannian metric, then $|\xi|$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ are well defined.

It is known from Refs. [8], and [5] that with the above conditions, the spectrum of P_ε in a small but fixed neighborhood of E in \mathbb{C} is discrete, when $h > 0, \varepsilon \geq 0$ are small enough. Moreover, this spectrum is contained in a horizontal band of size ε :

$$|\text{Im}(z)| \leq \mathcal{O}(\varepsilon). \quad (2.14)$$

Let $p = p_{\varepsilon=0}$, it is principal symbol of the selfadjoint unperturbed operator P and therefore real. And let $q = \frac{1}{i} \left(\frac{\partial}{\partial \varepsilon} \right)_{\varepsilon=0} p_\varepsilon$ and assume that q is a bounded analytic function on T^*M . We can write the principal symbol

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2). \quad (2.15)$$

We assume that p is completely integrable, i.e., there exists an analytic real valued function f , differentially independent of p such that $\{p, f\} = 0$ with respect to the Poisson bracket on T^*M . That means

$$F = (p, f) : T^*M \rightarrow \mathbb{R}^2 \quad (2.16)$$

is a momentum map. Then the space of regular leaves of F is foliated by Liouville Lagrangian invariant tori by the angle-action Theorem 2.20.

We assume also that

$$p^{-1}(E) \cap T^*M \text{ is connected,} \quad (2.17)$$

and the energy level E is regular for p , i.e., $dp \neq 0$ on $p^{-1}(E) \cap T^*M$. The energy space $p^{-1}(E)$ is decomposed into a singular foliation:

$$p^{-1}(E) \cap T^*M = \bigcup_{a \in J} \Lambda_a, \quad (2.18)$$

where J is assumed to be a compact interval, or, more generally, a connected graph with a finite number of vertices and of edges, see pp. 21-22 and 55 of Ref. [8].

We denote by S the set of vertices. For each $a \in J$, Λ_a is a connected compact subset invariant with respect to H_p . Moreover, if $a \in J \setminus S$, Λ_a is a invariant Lagrangian torus depending analytically on a . These tori are regular leaves corresponding regular values of F . Each edge of J can be identified with a bounded interval of \mathbb{R} and we have therefore a distance on J in the natural way.

We denote H_p the Hamiltonian vector field of p , defined by $\sigma(H_p, \cdot) = -dp(\cdot)$. For each $a \in J$, we define a compact interval in \mathbb{R} :

$$Q_\infty(a) = \left[\liminf_{T \rightarrow \infty} \lim_{\Lambda_a} Re \langle q \rangle_T, \limsup_{T \rightarrow \infty} \lim_{\Lambda_a} Re \langle q \rangle_T \right], \quad (2.19)$$

where $\langle q \rangle_T$, for $T > 0$, is the symmetric average time T of q along the H_p -flow, defined by

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp(tH_p) dt. \quad (2.20)$$

Then it is more detail than (2.14), the spectrum the of P_ε in the neighborhood of E in \mathbb{C} is located in the band

$$\text{Im}(\sigma(P_\varepsilon) \cap \{z \in \mathbb{C} : |Rez - E| \leq \delta\}) \subseteq \varepsilon \left[\inf_{a \in J} Q_\infty(a) - o(1), \sup_{a \in J} Q_\infty(a) + o(1) \right], \quad (2.21)$$

when $\varepsilon, h, \delta \rightarrow 0$ (see Ref. [8]).

Each invariant Lagrangian torus Λ_a , with $a \in J \setminus S$, locally can be embedded in a Lagrangian foliation of H_p -invariant tori. By the angle-action Theorem 2.20, there are analytic local angle-action coordinates $\kappa = (x, \xi)$ on an open neighborhood V of Λ_a in T^*M ,

$$\kappa = (x, \xi) : V \rightarrow \mathbb{T}^2 \times A, \quad (2.22)$$

with A is an open neighborhood of some $\xi_a \in \mathbb{R}^2$, such that Λ_a is symplectically identified with $\mathbb{T}^2 \times \{\xi_a\}$ by κ , and p becomes a function of action variables ξ ,

$$p \circ \kappa^{-1} = p(\xi) = p(\xi_1, \xi_2), \quad \xi \in A. \quad (2.23)$$

Let $\Lambda \subset V$ be an arbitrary invariant Lagrangian torus (close to Λ_a) and suppose that by κ , it is symplectically sent to the torus $\mathbb{T}^2 \times \{\xi\}$, denoted by Λ_ξ , with some $\xi \in A$. We introduce the following notation used throughout this paper

$$\Lambda \simeq \Lambda_\xi. \quad (2.24)$$

Then the frequency of the torus Λ (also of Λ_ξ) is defined by

$$\omega(\xi) = \frac{\partial p}{\partial \xi}(\xi) = \left(\frac{\partial p}{\partial \xi_1}(\xi), \frac{\partial p}{\partial \xi_2}(\xi) \right), \quad \xi \in A. \quad (2.25)$$

In particular, the frequency Λ_a is $\omega(\xi_a) = \frac{\partial p}{\partial \xi}(\xi_a)$. It knows that ω depends analytically on $\xi \in A$. In particular, the restriction $\omega(\xi_a)$ depends analytically on ξ_a , when $a \in J \setminus S$. We will assume that the function $a \mapsto \omega(\xi_a)$ is not identically constant on any connected component of $J \setminus S$.

From now, for simplicity, we will assume that q is **real valued** (in the general case, simply replace q by its real part $\text{Re}(q)$).

We define the average of q on the torus Λ , with respect to the natural Liouville measure on Λ , denoted by $\langle q \rangle_{\Lambda_\xi}$, as following

$$\langle q \rangle_\Lambda = \int_\Lambda q. \quad (2.26)$$

Remark 2.3. In the action-angle coordinates (x, ξ) given by (2.22), we have

$$\langle q \rangle_\Lambda = \langle q \rangle_{\Lambda_\xi} = \langle q \rangle(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} q(x, \xi) dx, \quad \xi \in A. \quad (2.27)$$

In particular, $\langle q \rangle_{\Lambda_a} = \langle q \rangle(\xi_a)$.

Remark 2.4 (see pp. 56-57 of Ref. [8]). For $a \in J \setminus S$, if $\omega(a) \notin \mathbb{Q}$, that means the frequency $\omega(a)$ is non resonant, then along the torus Λ_a , the Hamiltonian flow of p is ergodic. Hence the limit of $\langle q \rangle_T$, when $T \rightarrow \infty$ exists, and is equals to the space average of q over the torus, $\langle q \rangle_{\Lambda_a}$. Therefore we have

$$Q_\infty(a) = \{\langle q \rangle_{\Lambda_a}\}.$$

It is true that $\langle q \rangle_{\Lambda_a}$ depends analytically on $a \in J \setminus S$, and we assume that it can be extended continuously on J . Furthermore, we assume that the function $a \mapsto \langle q \rangle_{\Lambda_a} = \langle q \rangle(\xi_a)$ is not identically constant on any connected component of $J \setminus S$.

Assume furthermore that the differentials of the functions $p(\xi)$ given in (2.23), and of $\langle q \rangle$ given in (2.27) are \mathbb{R} -linearly independent when $\xi = \xi_a$. Then $\langle q \rangle$ and p are in involution in the neighborhood V of Λ_a , due to the fact that $\langle q \rangle$ is invariant under the flow of p .

2.1.2 Asymptotic expansion of eigenvalues

The asymptotic spectral theory (see Refs [5]- [8]) allows us to give an asymptotic description of all the eigenvalues of P_ε in some adapted small complex windows of the spectral band, which are associated with Diophantine tori in the phase space. The force of the perturbation ε is small and can be dependent or independent of the classical parameter h . In our work we present the result in the case when ε is sufficiently small, dependent on h , and in the following regime

$$h \ll \varepsilon = \mathcal{O}(h^\delta), \quad (2.28)$$

where $\delta > 0$ is some number small enough but fixed. In this case, spectral results are related to (h, ε) -dependent small windows.

Definition 2.5. *Let $\alpha > 0$, $d > 0$, and $\Lambda \simeq \Lambda_\xi$ be a H_p -invariant Lagrangian torus, as in (2.24). We say that Λ is (α, d) -Diophantine if its frequency $\omega(\xi)$, defined in (2.25), satisfies*

$$\omega(\xi) \in D_{\alpha,d} = \left\{ \omega \in \mathbb{R}^2 \mid |\langle \omega, k \rangle| \geq \frac{\alpha}{|k|^{1+d}}, \forall k \in \mathbb{Z}^2 \setminus \{0\} \right\}. \quad (2.29)$$

If (2.29) holds for some $\alpha > 0$, and $d > 0$, we say that the torus Λ (also its frequency) is uniformly Diophantine.

Notice also that when $d > 0$ is fixed, the Diophantine property (for some $\alpha > 0$) of Λ is independent of selected action-angle coordinates, see [10]. If Λ is (α, d) -Diophantine, then its frequency must be irrational.

It is known that the set $D_{\alpha,d}$ is a closed set with closed half-line structure. When we take α to be sufficiently small, it is a nowhere dense set but with no isolated points. Moreover, its measure tends to full measure as α tends to 0: the measure of its complement is of order $\mathcal{O}(\alpha)$. On the other hand, the trace of $D_{\alpha,d}$ on the unit sphere is a Cantor set. See Refs. [15], [12], and [9].

Definition 2.6. *For some $\alpha > 0$ and some $d > 0$, we define the set of good values associated with an energy level E , denoted by $\mathcal{G}(\alpha, d, E)$, obtained from $\cup_{a \in J} Q_\infty(a)$ by removing the following set of bad values $\mathcal{B}(\alpha, d, E)$:*

$$\begin{aligned} \mathcal{B}(\alpha, d, E) = & \left(\bigcup_{\text{dist}(a,S) < \alpha} Q_\infty(a) \right) \cup \left(\bigcup_{a \in J \setminus S: \omega(\xi_a) \text{ is not } (\alpha,d)\text{-Diophantine}} Q_\infty(a) \right) \\ & \cup \left(\bigcup_{a \in J \setminus S: |d\langle q \rangle(\xi_a)| < \alpha} Q_\infty(a) \right) \cup \left(\bigcup_{a \in J \setminus S: |\omega'(\xi_a)| < \alpha} Q_\infty(a) \right). \end{aligned}$$

Remark 2.7. (i) When $d > 0$ is kept fixed, the measure of the set of bad values $\mathcal{B}(\alpha, d, E)$ in $\cup_{a \in J} Q_\infty(a)$ (and in $\langle q \rangle_{\Lambda_a}(J)$) is small together with α , is $\mathcal{O}(\alpha)$, when

$\alpha > 0$ is small enough, provided that the measure of

$$\left(\bigcup_{a \in J \setminus S: \omega(\xi_a) \in \mathbb{Q}} Q_\infty(a) \right) \cup \left(\bigcup_{a \in S} Q_\infty(a) \right) \quad (2.30)$$

is sufficiently small, depending on α (see Ref. [8]).

- (ii) Let $G \in \mathcal{G}(\alpha, d, E)$ be a good value, then by definition of $\mathcal{B}(\alpha, d, E)$ and Remark 2.4, there are a finite number of corresponding (α, d) -Diophantine tori $\Lambda_{a_1}, \dots, \Lambda_{a_L}$, with $L \in \mathbb{N}^*$ and $\{a_1, \dots, a_L\} \subset J \setminus S$, in the energy space $p^{-1}(E) \cap T^*M$, such that the pre-image

$$\langle q \rangle^{-1}(G) = \{\Lambda_{a_1}, \dots, \Lambda_{a_L}\}.$$

In this way, when G varies in $\mathcal{G}(\alpha, d, E)$, we obtain a Cantor family of (α, d) -Diophantine invariant tori in the phase space satisfying $\{p = E, \langle q \rangle = G\}$.

When $G \in \mathcal{G}(\alpha, d, E)$ is a good value, we define in the horizontal band of size ε of complex plan, given in (2.21), a suitable window of size $\mathcal{O}(h^\delta) \times \mathcal{O}(\varepsilon h^\delta)$, around the good center $E + iG$, called *good rectangle*,

$$R^{(E,G)}(\varepsilon, h) = (E + i\varepsilon G) + \left[-\frac{h^\delta}{\mathcal{O}(1)}, \frac{h^\delta}{\mathcal{O}(1)} \right] + i\varepsilon \left[-\frac{h^\delta}{\mathcal{O}(1)}, \frac{h^\delta}{\mathcal{O}(1)} \right]. \quad (2.31)$$

Now let $G \in \mathcal{G}(\alpha, d, E)$ be a good value. As in Remark 2.7, there exists L elements in pre-image of G by $\langle q \rangle$. Spectral results related to each of these elements are similar. Therefore, to simplify to announce the results (in Theorem 2.9), we shall assume that $L = 1$ and we write

$$\langle q \rangle^{-1}(G) = \Lambda_a \subset p^{-1}(E) \cap T^*M, \quad a \in J \setminus S, \quad (2.32)$$

Note that this hypothesis can be achieved if we assume that the function $\langle q \rangle$ is injective on $J \setminus S$.

Definition 2.8 (Refs. [25], [1], and [34]). *Let E be a symplectic space and let $\Lambda(E)$ be his Lagrangian Grassmannian (which is set of all Lagrangian subspaces of E). We consider a bundle B in E over the circle or a compact interval provided with a Lagrangian subbundle called vertical. Let $\lambda(t)$ be a section of $\Lambda(B)$ which is transverse to the vertical edges of the interval in the case where the base is an interval. The Maslov index of $\lambda(t)$ is the intersection number of this curve with the singular cycle of Lagrangians which do not cut transversely the vertical subbundle.*

Let Λ_a , $a \in J \setminus S$ be an invariant Lagrangian torus and let κ be the action-angle local coordinates in (2.22). The fundamental cycles (γ_1, γ_2) of Λ_a are defined by

$$\gamma_j = \kappa^{-1}(\{(x, \xi) \in T^*\mathbb{T}^2 : x_j = 0, \xi = \xi_a\}), \quad j = 1, 2.$$

Then we note $\eta \in \mathbb{Z}^2$ the Maslov indices and $S \in \mathbb{R}^2$ the action integrals of these fundamental cycles,

$$S = (S_1, S_2) = \left(\int_{\gamma_1} \theta, \int_{\gamma_2} \theta \right), \quad (2.33)$$

where θ is the Liouville 1-form on T^*M .

We recall here spectral asymptotic results for the stand case at the energy level $E = 0$, given by [8]. However these results can be immediately generalized for any energy lever E by a translation, that we will carry out further in Sec. 2.2.

Theorem 2.9 ([8]). *For $E = 0$ and assume that the action-angle coordinates κ in (2.22) send Λ_a to the zero section $\mathbb{T}^2 \times \{\xi_a = 0\} \in T^*\mathbb{T}^2$. Suppose that P_ε is an operator with principal symbol (2.15) which satisfies Assumptions 2.1.1. Assume that $h \ll \varepsilon = \mathcal{O}(h^\delta)$ for $0 < \delta < 1$. Let $G \in \mathcal{G}(\alpha, d, 0)$ be a good value as Defenition 2.6, and assume that (2.32) is true. Then the eigenvalues μ of P_ε with multiplicity in the good rectangle $R^{(0,G)}(\varepsilon, h)$ of the form (2.31) have the following expression:*

$$\mu = P^{(\infty)}\left(h\left(k - \frac{\eta}{4}\right) - \frac{S}{2\pi}; \varepsilon, h\right) + \mathcal{O}(h^\infty), k \in \mathbb{Z}^2, \quad (2.34)$$

where $P^{(\infty)}(\xi; \varepsilon, h)$ is a smooth function of ξ evolving in a neighborhood of $(0, \mathbb{R}^2)$ and ε, h in neighborhoods of $(0, \mathbb{R})$. Moreover $P^{(\infty)}(\xi; \varepsilon, h)$ is real valued for $\varepsilon = 0$, admits the following polynomial asymptotic expansion in (ξ, ε, h) for the C^∞ -topology:

$$P^{(\infty)}(\xi; \varepsilon, h) \sim \sum_{\alpha, j, k} C_{\alpha j k} \xi^\alpha \varepsilon^j h^k. \quad (2.35)$$

Particularly $P^{(\infty)}$ is classical in the space of symbols with h -leading term:

$$p_0^{(\infty)}(\xi, \varepsilon) = p(\xi) + i\varepsilon \langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2). \quad (2.36)$$

Here p, q are the expressions of p, q in the action-angle variables near Λ_a , given by (2.22), and $\langle q \rangle$ is the average of q on tori, given in (2.27).

Remark 2.10. We can write the total symbol $P^{(\infty)}$ in the reduce form:

$$P^{(\infty)}(\xi; \varepsilon, h) = p(\xi) + i\varepsilon \langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(h), \quad (2.37)$$

uniformly for ε and h small. Notice that $dp, d\langle q \rangle$ are linearly independent in $\xi = 0$, and hence in the regime $h \ll \varepsilon$ we can show that the function $P^{(\infty)}$ is a local diffeomorphism from a neighborhood $B(0, r)$, $r > 0$, of $\xi = 0 \in \mathbb{R}^2$ into its image, which is in a $\mathcal{O}(\varepsilon)$ -horizontal band of the complex plane and covers the good rectangle (see Ref. [35]).

Remark 2.11. By Theorem 2.9, it is true that the eigenvalues of P_ε in the good rectangle $R^{(0,G)}(\varepsilon, h)$ form a deformed lattice. In the case $L \geq 1$, then the spectrum of P_ε in $R^{(0,G)}(\varepsilon, h)$ are union of L such deformed lattices.

2.2 Spectral asymptotic pseudo-lattice and its monodromy

We are considering the operator P_ε satisfying Assumptions Sec. 2.1.1 and in the regime (2.28). We first apply the results of the stand case, stated in Theorem 2.9, to given an asymptotic expansion of the eigenvalues of P_ε in any good rectangle $R^{(E,G)}(\varepsilon, h)$ of the spectral band. After, we will give a brief construction to show that the spectrum satisfies all conditions of an asymptotic pseudo-lattice. We refer to [35] for a similar construction with more detail.

We notice that $\langle q \rangle = \langle q \rangle_\Lambda$, the average of q over the regular H_p -invariant Lagrangian torus Λ , given in (2.26), can be seen as a constant function on Λ . In this way $\langle q \rangle$ well defines a analytic function on the union of regular invariant Lagrangian tori. It is true that the fibration of all the regular invariant Lagrangian tori Λ , given by the momentum map F , fills almost completely the phase space T^*M . So we assume further that the function $\langle q \rangle$ can be extended smoothly on the whole phase space T^*M . Moreover, we assume that the differentials dp and $d\langle q \rangle$ are \mathbb{R} -linearly independent almost everywhere on T^*M . It is clear that p and $\langle q \rangle$ commute in neighborhood of each regular torus Λ_a . So we can see $(p, \langle q \rangle)$ as a completely integrable system, and we take now the momentum map $F = (p, f)$ in (2.16), with $f = \langle q \rangle$.

We assume moreover that the map F is proper and has connected fibers. This ensures that all regular fibres of F are invariant Lagrangian tori-the Liouville tori. Denote U_r the set of all regular values of F , and let U be an open subset of U_r with compact closure.

We assume further that all assumptions of Sec. 2.1.1 are true for any energy level E , taken in a bounded interval of \mathbb{R} . Of course, we can take E in the interval $Pr(U)$, where Pr is the projection on the (real) horizontal axis.

Let a point $c \in U$. Then by the angle-action Theorem 2.20, we have local angle-action variables on a neighborhood of the torus $\Lambda_c = F^{-1}(c)$ in T^*M as in (2.22): there exists a sufficiently small open neighborhood $U^c \subset U$ of c , a symplectomorphism $\kappa = (x, \xi) : V \rightarrow \mathbb{T}^n \times A$, here $V := F^{-1}(U^c)$ and $A \subset \mathbb{R}^2$ is a small open set, a diffeomorphism $\varphi = (\varphi_1, \varphi_2) : A \rightarrow U^c$ such that $F \circ \kappa^{-1}(x, \xi) = \varphi(\xi)$, for all $x \in \mathbb{T}^2$, $\xi \in A$.

Therefore, as in (2.23) and (2.27), the expressions of the functions p and $\langle q \rangle$ in the above action-angle coordinates is dependent only on action variables,

$$p(\xi) = \varphi_1(\xi), \langle q \rangle(\xi) = \varphi_2(\xi), \xi \in A.$$

The fact that the horizontal spectral band is of size $\mathcal{O}(\varepsilon)$ suggests us introducing the function

$$\begin{aligned} \chi : \mathbb{R}^2 \ni u = (u_1, u_2) &\mapsto \chi_u = (u_1, \varepsilon u_2) \in \mathbb{R}^2 \\ &\cong u_1 + i\varepsilon u_2 \in \mathbb{C}, \end{aligned} \tag{2.38}$$

in which we identify \mathbb{C} with \mathbb{R}^2 . We denote

$$U^c(\varepsilon) = \chi(U^c), U(\varepsilon) = \chi(U). \tag{2.39}$$

Now, with any point $a = (E, G) \in U^c$ such that G is a good value, i.e., $G \in \mathcal{G}(\alpha, d, E)$, as in Definition 2.6. We notice also that with the assumptions of F , the condition (2.32) is valid ($L = 1$). Then the corresponding torus $\Lambda_a = F^{-1}(a)$ is (α, d) -Diophantine, as discussed in Remark 2.7. We suppose that by κ , $\Lambda_a \simeq \Lambda_{\xi_a}$, with $\xi_a = \varphi^{-1}(a) \in A$. Let $\eta \in \mathbb{Z}^2$ be the Maslov indices, and $S \in \mathbb{R}^2$ be the action integrals of the fundamental cycles of Λ_a .

Noticing that $\sigma(P_\varepsilon) = \sigma(P_\varepsilon - \chi_a) + \chi_a$ and applying Theorem 2.9 (in the stand case) for the operator $(P_\varepsilon - \chi_a)$ with respect to the good rectangle $R^{(0,G)}(\varepsilon, h)$, we obtain easily the asymptotic eigenvalues of P_ε in the good rectangle $R^{(E,G)}(\varepsilon, h)$, as following.

All eigenvalues μ of P_ε in the good rectangle $R^{(E,G)}(\varepsilon, h)$ defined by (2.31), with modulo $\mathcal{O}(h^\infty)$, are *micro-locally* given by

$$\sigma(P_\varepsilon) \cap R^{(E,G)}(\varepsilon, h) \ni \mu = P\left(\xi_a + h(k - \frac{\eta}{4}) - \frac{S}{2\pi}; \varepsilon, h\right) + \mathcal{O}(h^\infty), k \in \mathbb{Z}^2, \quad (2.40)$$

where $P(\xi; \varepsilon, h)$ (for ease the notation, writing P instead of $P^{(\infty)}$, given in Theorem 2.9) is a smooth function of ξ in a neighborhood of (ξ_a, \mathbb{R}^2) and of ε, h in neighborhoods of $(0, \mathbb{R})$. Moreover, $P(\xi; \varepsilon, h)$ admits an asymptotic expansion in (ξ, ε, h) of the form (2.35), and in particular the h -leading term of P is of the form (2.36):

$$p_0(\xi, \varepsilon) = p(\xi) + i\varepsilon\langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2) \quad (2.41)$$

$$= \varphi_1(\xi) + i\varepsilon\varphi_2(\xi) + \mathcal{O}(\varepsilon^2). \quad (2.42)$$

Remark 2.12.

- (i) The previous result shows that the eigenvalues of P_ε in a good rectangle form a deformed *micro-lattice*. It's image of a square lattice of $h\mathbb{Z}^2$ by a local diffeomorphism. Moreover, we can show that such a micro-lattice has a horizontal spacing $\mathcal{O}(h)$ and a vertical spacing $\mathcal{O}(\varepsilon h)$, see [35].
- (ii) In this paper, the vocabulary *micro* is used for some objects, which are related to a very small domain depending on h .

According the definition of an *asymptotic pseudo-lattice* given in [35] (recalled in Definition 2.15), and with the help of the micro-locally asymptotic spectrum (2.40), the next construction is to show the following statement.

Theorem 2.13. *The spectrum of P_ε on the domain $U(\varepsilon)$ is an asymptotic pseudo-lattice.*

As a classical property, it is true that the difference between the action integral, with factor $\frac{1}{2\pi}$, and the action coordinates are locally constant on every regular tori $\Lambda_a \subset V$ (see Ref. [35]):

$$\frac{S}{2\pi} - \xi_a := \tau_c \in \mathbb{R}^2, \quad (2.43)$$

is locally constant in $c \in U_r$.

From Remark 2.10, it is true that Eq. (2.40) provides a bijective correspondence between the eigenvalues in the good rectangle $R^{(E,G)}(\varepsilon, h)$ and hk in a part of $h\mathbb{Z}^2$. Moreover, this correspondence is given by a smooth local diffeomorphism in $E \in \mathbb{C}$, denoted by $f = f(\mu; \varepsilon, h)$, which sends micro-locally $R^{(E,G)}(\varepsilon, h)$ to its image, denoted by $E(a, \varepsilon, h)$ (which is close to $\frac{S}{2\pi}$), such that $\mu \in \sigma(P_\varepsilon) \cap R^{(E,G)}(\varepsilon, h)$ is sent to $hk \in h\mathbb{Z}^2$, modulo $\mathcal{O}(h^\infty)$:

$$\begin{aligned} f = f(\mu, \varepsilon; h) &= \tau_c + h\frac{\eta}{4} + P^{-1}(\mu) \\ f(R^{(E,G)}(\varepsilon, h)) &= E(a, \varepsilon, h) \\ \sigma(P_\varepsilon) \cap R^{(E,G)}(\varepsilon, h) \ni \mu &\mapsto f(\mu, \varepsilon; h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty). \end{aligned} \quad (2.44)$$

We say that the map f in (2.44) is a *micro-chart* of the spectrum of P_ε on the good rectangle $R^{(E,G)}(\varepsilon, h)$. We shall analyze it.

Let $\tilde{f} = f \circ \chi$, then we have

$$\tilde{f} = \tau_c + h\frac{\eta}{4} + P^{-1} \circ \chi. \quad (2.45)$$

To analysis $P^{-1} \circ \chi$, we first discuss about its inverse, $\widehat{P} := \chi^{-1} \circ P$. It is obtained from P by dividing the imaginary part of P by ε . As P admits an asymptotic expansion in (ξ, ε, h) , so it is true that \widehat{P} admits an asymptotic expansion in $(\xi, \varepsilon, \frac{h}{\varepsilon})$ (here $h \ll \varepsilon$). Moreover, we can write \widehat{P} in the reduce form as below:

$$\begin{aligned} \widehat{P}(\xi, \varepsilon, h) &= \widehat{P}_0(\xi) + \mathcal{O}(\varepsilon) + \mathcal{O}\left(\frac{h}{\varepsilon}\right) \\ &= \widehat{P}_0(\xi) + \mathcal{O}\left(\varepsilon, \frac{h}{\varepsilon}\right), \end{aligned} \quad (2.46)$$

uniformly for h, ε small and $h \ll \varepsilon$, with $\widehat{P}_0(\xi) = \varphi_1(\xi) + i\varphi_2(\xi)$.

Proposition 2.14. (see [35]) *Let $\widehat{P} = \widehat{P}(\xi; X)$ a complex-valued smooth function of ξ near $0 \in \mathbb{R}^2$ and X near $0 \in \mathbb{R}^n$. Assume that \widehat{P} admits an asymptotic expansion in X near 0 of the form*

$$\widehat{P}(\xi; X) \sim \sum_{\alpha} C_{\alpha}(\xi) X^{\alpha},$$

with $C_{\alpha}(\xi)$ are smooth functions and $C_0(\xi) := \widehat{P}_0(\xi)$ is local diffeomorphism near $\xi = 0$.

Then, for $|X|$ small enough, \widehat{P} is also a smooth local diffeomorphism near $\xi = 0$ and its inverse admits an asymptotic expansion in X near 0 whose the first term is $(\widehat{P}_0)^{-1}$.

This proposition ensures that the map $\widehat{P}^{-1} = P^{-1} \circ \chi$ admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ whose first term is $(\widehat{P}_0)^{-1} = \varphi^{-1}$. So from (2.45), \tilde{f} admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ with the leading term

$$\tilde{f}_0 = \tau_c + \varphi^{-1}. \quad (2.47)$$

We notice that the leading term \tilde{f}_0 is a local diffeomorphism, completely defined on U^c . It does not depend on the selected good rectangle, is valid for any good value $a \in U^c$.

However, in the domain $U^c(\varepsilon)$, we have a lot of good rectangles. They could be disjoint when h is sufficiently small, and despite their density, not quite fill $U^c(\varepsilon)$. In fact, we know moreover from Remark 2.7 that the set of good values is a Cantor set, and outside a set of small measure. We have therefore locally a Cantor family of micro-charts with common leading term that is well locally defined. This family is called a *spectral pseudo-chart* on the domain $U^c(\varepsilon)$.

The above construction ensures that the spectrum of P_ε on the domain $U(\varepsilon)$ satisfies all conditions of a particular discrete lattice—an asymptotic pseudo-lattice, given in [35], that we recall here:

Definition 2.15. *Let U be a open subset of \mathbb{R}^2 with compact closure. We denote $U(\varepsilon)$ as in (2.39), and let $\Sigma(\varepsilon, h)$ (which depends on small h and ε) be a discrete subset of $U(\varepsilon)$. For h, ε small enough and in the regime $h \ll \varepsilon$, we say that $(\Sigma(\varepsilon, h), U(\varepsilon))$ is an asymptotic pseudo-lattice if: for any small parameter $\alpha > 0$, there exists a set of good values in \mathbb{R}^2 , denoted by $\mathcal{G}(\alpha)$, whose complement is of small measure together with α in the sense:*

$$|{}^C\mathcal{G}(\alpha) \cap I| \leq C\alpha |I|,$$

with a constant $C > 0$ for any domain $I \subset \mathbb{R}^2$.

For all $c \in U$, there exists a small open subset $U^c \subset U$ around c such that for every good value $a \in U^c \cap \mathcal{G}(\alpha)$, there is a adapted good rectangle $R^{(a)}(\varepsilon, h) \subset U^c(\varepsilon)$ of the form (2.31) (with $a = (E, G)$), and a smooth local diffeomorphism $f = f(\cdot; \varepsilon, h)$ which sends $R^{(a)}(\varepsilon, h)$ on its image, satisfying

$$\Sigma(\varepsilon, h) \cap R^{(a)}(\varepsilon, h) \ni \mu \mapsto f(\mu; \varepsilon, h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty). \quad (2.48)$$

Moreover, the function $\tilde{f} := f \circ \chi$, with χ defined by (2.38), admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ for the C^∞ -topology in a neighborhood of a , uniformly with respect to the parameters h and ε , such that its leading term \tilde{f}_0 is a diffeomorphism, independent of α , locally defined on the whole U^c and independent of the selected good values $a \in U^c$.

We also say that the couple $(f(\cdot; \varepsilon, h), R^{(a)}(\varepsilon, h))$ is a micro-chart, and the family of micro-charts $(f(\cdot; \varepsilon, h), R^{(a)}(\varepsilon, h))$, with all $a \in U^c \cap \mathcal{G}(\alpha)$, is a local pseudo-chart on $U^c(\varepsilon)$ of $(\Sigma(\varepsilon, h), U(\varepsilon))$.

Remark 2.16. The introduction of this discrete lattice aims to show that the monodromy that we will define is directly built from the spectrum of operators. If different operators have the same spectrum, then they have the same monodromy.

Now we can define the spectral monodromy of the operator P_ε on the domain $U(\varepsilon)$, denoted by $[\mathcal{M}_{sp}]$, as the monodromy of the asymptotic pseudo-lattice $(\sigma(P_\varepsilon), U(\varepsilon))$, as the following.

Let $\{U^j\}_{j \in \mathcal{J}}$, here \mathcal{J} is a finite index set, be an arbitrary (small enough) locally finite covering of U . Then the asymptotic pseudo-lattice $(\Sigma(\varepsilon, h), U(\varepsilon))$ can be covered by the associated local pseudo-charts $\{(f_j(\cdot; \varepsilon, h), U^j(\varepsilon))\}_{j \in \mathcal{J}}$, here $U^j(\varepsilon) = \chi(U^j)$. Note from Definition 2.15 that the leading terms $\tilde{f}_{j,0}(\cdot; \varepsilon, h)$ are well defined on whole U^j and we can see them as the charts of U . Analyzing transition maps, we have the following result.

Theorem 2.17 ([35]). *On each nonempty intersection $U^i \cap U^j \neq \emptyset$, $i, j \in \mathcal{J}$, there exists a unique integer linear map $M_{ij} \in GL(2, \mathbb{Z})$ (independent of h, ε) such that:*

$$d(\tilde{f}_{i,0} \circ (\tilde{f}_{j,0})^{-1}) = M_{ij}. \quad (2.49)$$

Then we define the class, denoted by $[\mathcal{M}_{sp}] \in \check{H}^1(U, GL(2, \mathbb{Z}))$, obtained from the 1-cocycle of $\{M_{ij}\}$ with modulo coboundary, in the Čech cohomology of U with values in the integer linear group $GL(2, \mathbb{Z})$. It is called the (linear) monodromy of the asymptotic pseudo-lattice $(\Sigma(\varepsilon, h), U(\varepsilon))$. It does not depend on the selected finite covering $\{U^j\}_{j \in \mathcal{J}}$.

We can also associate the class $[\mathcal{M}_{sp}]$ with its holonomy, that is a group morphism from the fundamental group $\pi_1(U)$ of U to the group $GL(2, \mathbb{Z})$, modulo conjugation. Their trivial property is equivalent.

Definition 2.18. *For $\varepsilon, h > 0$ sufficiently small such that $h \ll \varepsilon \leq h^\delta$, $0 < \delta < 1$, the spectral monodromy of the operator P_ε on the domain $U(\varepsilon)$ is the class $[\mathcal{M}_{sp}] \in \check{H}^1(U, GL(2, \mathbb{Z}))$.*

2.3 Spectral monodromy recovers the classical monodromy

Our considered operators P_ε in the first case are related to integrable classical systems. With the property of classical integrability, a geometrical invariant of such systems that obstructs the existence of global angle-action variables on the phase space, is known (see Refs. [21], [4], and [18]). That is the classical monodromy, defined correctly by Duistermaat [16]. We make here a relationship between the spectral monodromy and the classical monodromy.

Definition 2.19. *A completely integrable system on a symplectic manifold (W, σ) of dimension $2n$ ($n \geq 1$) is given n smooth real-valued functions f_1, \dots, f_n in involution with respect to the Poisson bracket generated from the symplectic form σ , whose differentials are almost everywhere linearly independent. In this case, the map*

$$F = (f_1, \dots, f_n) : M \rightarrow \mathbb{R}^n$$

is called a momentum map or a completely integrable system.

Let U be a open subset of the set of regular values of F . Then we have:

Theorem 2.20 (Angle-action theorem). *Let $c \in U$, and Λ_c be a compact regular leaf of the fiber $F^{-1}(c)$. Then there exists an open neighborhood $V = V^c$ of Λ_c in W such that $F|_V$ defines a smooth locally trivial fibre bundle onto an open neighborhood $U^c \subset U$ of c , whose fibres are invariant Lagrangian n -tori. Moreover, there exists a symplectic diffeomorphism $\kappa = \kappa^c$,*

$$\kappa = (x, \xi) : V \rightarrow \mathbb{T}^n \times A,$$

with $A = A^c \subset \mathbb{R}^n$ is an open subset, such that $F \circ \kappa^{-1}(x, \xi) = \varphi(\xi)$ for all $x \in \mathbb{T}^n$, and $\xi \in A$, and here $\varphi = \varphi^c : A \rightarrow \varphi(A) = U^c$ is a local diffeomorphism. We call (x, ξ) local angle-action variables near Λ_c and (V, κ) an local angle-action chart.

Notice that one chooses usually the local chart such that the torus Λ_c is sent by κ to the zero section $T^n \times \{0\}$. By this theorem, for every $a \in U^c$, then $\Lambda_a := F^{-1}(a) \cap V^c$ are invariant Lagrangian n -tori, called Liouville tori, and we have $\Lambda_a \simeq \Lambda_{\xi_a}$, with some $\xi_a \in A$.

Moreover, there exists an affine structure on the geometry of Liouville tori (see Refs. [16], and [27]). We assume for simplicity that F is proper and connected fibres. Then every regular fibre $\Lambda_c := F^{-1}(c)$, with $c \in U$, is a Liouville torus.

When U is assumed furthermore relatively compact subset and we cover it by an arbitrary small enough finite open covering $\{U^j\}_{j \in \mathcal{J}}$, here \mathcal{J} is a finite index set. Then the relatively compact subspace $X = F^{-1}(U)$ can also be covered by a finite covering of angle-action charts $\{(V^j, \kappa^j)\}_{j \in \mathcal{J}}$. Here $V^j = F^{-1}(U^j)$, $j \in \mathcal{J}$. It is classical that on the nonempty overlaps $V^i \cap V^j$, there is a locally constant matrix $M_{ij}^{cl} \in GL(2, \mathbb{Z})$, and a constant $C_{ij} \in \mathbb{R}^2$, with $i, j \in \mathcal{J}$, such that

$$(\varphi^i)^{-1} \circ \varphi^j(\cdot) = M_{ij}^{cl}(\cdot) + C_{ij}. \quad (2.50)$$

The non-triviality of M_{ij} leads surely to a obstruction of the existence of global action-angle coordinates, called classical monodromy. It is defined as the \mathbb{Z}^n -bundle $H_1(\Lambda_c, \mathbb{Z}) \rightarrow c \in U$, where $H_1(\Lambda_c, \mathbb{Z})$ is the first homology group of Λ_c , whose transition maps between trivializations are

$$\{{}^t(d((\varphi^i)^{-1} \circ \varphi^j))^{-1} = {}^t(M_{ij}^{cl})^{-1}\}.$$

The triviality of this monodromy is equivalent to the existence of global action variables on the space of Liouville tori. For more on this monodromy, we refer also to Ref. [27].

With $W = T^*M$ and $n = 2$, we apply now the result (2.50) to spectral pseudo-charts of the operator P_ε , and noticing the expression (2.47). It is true that on the nonempty intersections $U^i \cap U^j \neq \emptyset$, the corresponding local spectral pseudo-charts of $(\sigma(P_\varepsilon), U(\varepsilon))$ satisfy

$$d(\tilde{f}_{i,0} \circ (\tilde{f}_{j,0})^{-1}) = M_{ij}^{cl} \in GL(2, \mathbb{Z}). \quad (2.51)$$

This result, due to classical results, asserts again the integer linear structure related to $(\sigma(P_\varepsilon), U(\varepsilon))$. It is completely independent of the result of Theorem 2.17. Therefore, we can state that in dimension 2, the classical monodromy of an integrable system is

completely identified with the spectral monodromy of small non-selfadjoint perturbations of selfadjoint classical operators, accepting this integrable system as the leading term of the unperturbed part. And moreover:

Theorem 2.21. *The spectral monodromy of P_ε is the adjoint of the classical monodromy.*

This result is the same one of paper [35], but the proposed operators here are in a more general context.

3 Spectral monodromy in the quasi-integrable case

In this section we are studying the spectral monodromy of small non-selfadjoint perturbations of selfadjoint classical operators in 2 dimension in the case when the leading symbol of the unperturbed part is quasi-integrable, together with a globally isoenergetic condition.

A radical general assumption in the spectral asymptotic construction of such operators from a h -dependent complex window, given in [8], is that the real energy surface at certain level of the unperturbed leading symbol possesses several Hamiltonian flow invariant Lagrangian tori, satisfying a uniformly Diophantine condition.

In the first case when the unperturbed principal symbol is completely integrable, this assumption is ensured as we known in Sec. 2.1. However, it still satisfactory when this symbol is close to a completely integrable one, according the classical Kolmogorov-Arnold-Moser (KAM) theory (see Refs. [3], [2], [15], and [12]).

We will give results about the asymptotic spectrum, similarly as in the first case, to show that the spectrum of such operators should be an asymptotic pseudo-lattice on the spectral band. Then we can define its spectral monodromy.

3.1 Quasi-periodic flows of quasi-integrable systems

Classical KAM theory allows to treat perturbations of a completely integrable Hamiltonian system. Under an isoenergetic condition, this theory proves the persistence of invariant Lagrangian tori, called KAM tori, on which the classical flow of the unperturbed system stills quasi-periodic with Diophantine constant frequencies. Moreover, the union of these KAM tori is a nowhere dense set, with complement of small measure in the phase space.

We consider a perturbed Hamiltonian that is close to a completely integrable (non-degenerate) one:

$$p_\lambda = p + \lambda p_1, \quad 0 < \lambda \ll 1, \quad (3.52)$$

where p and p_1 are holomorphic bounded Hamiltonians in a tubular neighborhood of T^*M , real on T^*M , and moreover p is assumed to be a completely integrable Hamiltonian system, as in Sec. 2.1.

Let $d > 0$ fixed, and let Λ_a be a H_p -invariant uniformly Diophantine Lagrangian torus in the energy space $p^{-1}(E)$ as Definition 2.5 with some $\alpha > 0$. In the angle-action coordinates (x, ξ) on a neighborhood V of Λ_a , given in (2.22), the function p_λ becomes:

$$p_\lambda \circ \kappa^{-1} = p_\lambda(x, \xi) = p(\xi) + \lambda p_1(x, \xi).$$

The Hamiltonian flow of p on a H_p -invariant Lagrangian torus $\Lambda \subset V$ (close to Λ_a), $\Lambda \simeq \Lambda_\xi = \mathbb{T}^2 \times \{\xi\}$, is quasi-periodic of the constant Hamiltonian vector field

$$H_p(x, \xi) = \omega_1(\xi) \frac{\partial}{\partial x_1} + \omega_2(\xi) \frac{\partial}{\partial x_2}, \quad x \in \mathbb{T}^2, \quad \xi \in A, \quad (3.53)$$

with the frequency $\omega(\xi)$, given by (2.25).

In particular, the frequency of $\Lambda_a \simeq \Lambda_{\xi_a}$ is $\omega(\xi_a) = \frac{\partial p}{\partial \xi}(\xi_a)$ satisfying the Diophantine condition (2.29), for some $\alpha > 0$.

Definition 3.1. *We say that p is (Kolmogorov) local non-degenerate (on V) if the isoenergetic condition holds in the sense that the local frequency map $\omega : A \rightarrow \mathbb{R}^2$, defined by (2.25), is a diffeomorphism onto its image.*

In fact, this condition is equivalent to

$$\det \frac{\partial \omega}{\partial \xi} = \det \left(\frac{\partial^2 p}{\partial \xi^2} \right) \neq 0 \text{ on } A,$$

and it means that H_p -invariant Lagrangian tori near Λ_a can (locally) be parametrized by their frequencies.

Let $\Omega = \omega(A)$ be the open range of the frequency map ω . Let $\Omega_{\alpha,d} \subset \Omega$ be the subset of frequencies which satisfy the Diophantine condition (2.29) and whose distance to the boundary of Ω is at least equal to α . It is known that the set $\Omega_{\alpha,d}$ is a Cantor set (closed, perfect and nowhere dense) of full measure for sufficiently small α . The measure of $\Omega \setminus \Omega_{\alpha,d}$ is $\mathcal{O}(\alpha)$, which tends to zero as $\alpha \downarrow 0$, see Refs. [15], and [12]. Finally, we define the subset

$$A_{\alpha,d} = \omega^{-1}(\Omega_{\alpha,d}) \subset A.$$

It is true that $A_{\alpha,d}$ is also a Cantor set of full measure. The measure of the complement subset $A \setminus A_{\alpha,d}$ is of order $\mathcal{O}(\alpha)$ as $\alpha \downarrow 0$, see [9]. The intersection of $p^{-1}(E)$ with $\mathbb{T}^2 \times A$ is of the form $\mathbb{T}^2 \times \Gamma_a$, with a some curve denoted by Γ_a in A , passing through ξ_a .

Now for α small enough, we have the quasi-periodic stability of the Diophantine invariant Lagrangian tori in $\mathbb{T}^2 \times A_{\alpha,d}$, as the following theorem. It is combined from the different known versions of the classical KAM theorem, see Refs. [11], [14], [13], and [15].

Theorem 3.2. *Assume that p is local non-degenerate as Definition 3.1. Let $d > 0$ fixed and $\alpha > 0$ be small enough. Assume that $0 < \lambda \ll \alpha^2$. Then there exists a map $\Phi_\lambda : \mathbb{T}^2 \times A \rightarrow \mathbb{T}^2 \times A$ with the following properties:*

1. Φ_λ , depending analytically on λ , is a C^∞ -diffeomorphism onto its image, close to the identity map in the C^∞ -topology.
2. For each $\xi \in A$, the invariant Lagrangian torus $\Lambda_\xi = \mathbb{T}^2 \times \{\xi\}$ is sent, by Φ_λ , to $\Phi_\lambda(\Lambda_\xi)$ which is a Lagrangian torus, (close to Λ_ξ), denoted by Λ_{ξ_λ} , and of the form $\Lambda_{\xi_\lambda} = \mathbb{T}^2 \times \{\xi_\lambda\}$, with some ξ_λ in a certain open subset $A_\lambda \subset A$, induced by Φ_λ .

Moreover, if $\xi \in A_{\alpha,d}$, then Λ_{ξ_λ} , with ξ_λ in a certain subset $A_{\alpha,d,\lambda} \subset A_\lambda$, is still uniformly Diophantine H_{p_λ} -invariant torus, called a local KAM torus. The restricted map $\Phi_\lambda|_{\Lambda_\xi}$ on each Diophantine Lagrangian torus Λ_ξ , with $\xi \in A_{\alpha,d}$, conjugates the Hamiltonian vector field $H_p|_{\Lambda_\xi} = H_p(x, \xi)$, given in (3.53), to the Hamiltonian vector field $H_{p_\lambda}|_{\Lambda_{\xi_\lambda}}$, i.e., $\Phi_\lambda|_{\Lambda_\xi} * H_p = H_{p_\lambda}$.

In particular, if $\xi \in \Gamma_a$, then the torus $\Lambda_{\xi_\lambda} \subset p_\lambda^{-1}(E) \cap \mathbb{T}^2 \times \Gamma_{a,\lambda}$, with a certain curve $\Gamma_{a,\lambda} \subset A_\lambda$. Moreover, when $\xi \in \Gamma_a \cap A_{\alpha,d}$, the Liouville measure of the complement of the union of the KAM tori Λ_{ξ_λ} , in $p_\lambda^{-1}(E)$, is of order $\mathcal{O}(\alpha)$.

Note that in the previous theorem, we can see ξ_λ as a smooth function of $\xi \in A$ and with the parameter λ . From this theorem, we obtain a Cantor family of positive measure of KAM tori, that is $A_{\alpha,d,\lambda}$, on each of which the Hamiltonian flow of the perturbed system p_λ is quasi-periodic of constant vector field.

Remark 3.3. We would to cite a very interesting paper [9]. It attested that the unicity of KAM tori is valid on a subset of full measure of Diophantine Liouville tori. By eliminating a measure zero set from $A_{\alpha,d}$, one defines a subset $A_{\alpha,d}^* \subseteq A_{\alpha,d}$ as the set of density points of $A_{\alpha,d}$, as following. Let $D_{\alpha,d}^* \subseteq D_{\alpha,d}$ be the subset of density points of $D_{\alpha,d}$, and then let $A_{\alpha,d}^* = \omega^{-1}(D_{\alpha,d}^* \cap \Omega_{\alpha,d})$. The subset $A_{\alpha,d}^*$ has all similar properties than ones of $A_{\alpha,d}$, see [9]. The unicity here in the sense that the local conjugacy map Φ_λ given in Theorem 3.2, after restriction to $\mathbb{T}^2 \times A_{\alpha,d}^*$ is unique up to a torus translation. This unicity ensures the unicity of corresponding KAM tori in the phase space, that we shall discuss in Sec. 3.3.

3.2 Spectral monodromy of $P_{\varepsilon,\lambda}$

In this section, we use again notation of Sec. 2.2.

Let p_1 be an analytic function in a tubular neighborhood of T^*M , real on the real domain, with $p_1(x, \xi) = \mathcal{O}(m(\text{Re}(x, \xi)))$ in the case when $M = \mathbb{R}^2$, and $p_1(x, \xi) = \mathcal{O}(\langle \xi \rangle^m)$ in the manifold case.

Let

$$P_{\varepsilon,\lambda}, \quad 0 < \lambda \ll 1, \quad h \ll \varepsilon = \mathcal{O}(h^\delta), \quad \text{here } 0 < \delta < 1, \quad (3.54)$$

be a classical operator that is small perturbation of the selfadjoint operator $P_\lambda := P_{\varepsilon=0,\lambda}$, and with the h -leading term of the form

$$p_{\varepsilon,\lambda} = p_\lambda + i\varepsilon q + \mathcal{O}(\varepsilon^2), \quad \text{with } p_\lambda = p + \lambda p_1. \quad (3.55)$$

Here we assume that p, q are symbols satisfying all assumptions of Sec. 2.2, and moreover p is globally non-degenerate, as the following.

Let $X = F^{-1}(U)$, where the momentum map F and the set U are given in Sec. 2.2. Then the map $F|_X : X \rightarrow U$ defines a smooth locally trivial bundle, whose fibres are Liouville invariant Lagrangian tori. From the angle-action Theorem 2.20, we can cover X by an atlas of angle-action charts $\{(V^c, \kappa^c)\}_{c \in U}$.

Definition 3.4. *We say that p is globally non-degenerate (on X) if for such an arbitrary atlas of X , p is local non-degenerate on every local angle-action chart (V^c, κ^c) , according Definition 3.1.*

We shall introduce a set, associated with an energy level E of p_λ , which is similar to the set of good values, given in Definition 2.6 of the integrable case.

Let $a = (E, G)$ be a point in U such that G is a good value in $\mathcal{G}(\alpha, d, E)$, as in Definition 2.6. Then the corresponding torus $\Lambda_a \simeq \Lambda_{\xi_a}$, with $\xi_a \in \Gamma_a$, satisfies the (α, d) -Diophantine condition (2.29).

We assume moreover $0 < \lambda \ll \alpha^2$. Then by Theorem 3.2, there exists a smooth Cantor family of KAM tori close to Λ_{ξ_a} , $\Lambda_{\xi_\lambda} = \mathbb{T}^2 \times \{\xi_\lambda\}$, with $\xi_\lambda \in A_{\alpha, d, \lambda}$, on which the H_{p_λ} -flow is quasi periodic of a uniformly Diophantine constant frequency, denoted by $\omega_\lambda(\xi_\lambda)$. Therefore, over these KAM tori Λ_{ξ_λ} , p_λ become a function of only ξ_λ :

$$p_\lambda = p_\lambda(\xi_\lambda), \xi_\lambda \in A_{\alpha, d, \lambda}.$$

In particular, when $\xi \in \Gamma_a \cap A_{\alpha, d}$, then $\Lambda_{\xi_\lambda} \subset p_\lambda^{-1}(E) \cap \mathbb{T}^2 \times (\Gamma_{a, \lambda} \cap A_{\alpha, d, \lambda})$, and with $\xi_\lambda \in \Gamma_{a, \lambda} \cap A_{\alpha, d, \lambda}$.

Similarly as in (2.26), we define locally a smooth function $\langle q \rangle_{\Lambda_{\xi_\lambda}}$ of $\xi_\lambda \in A_\lambda$, denoted by $\langle q \rangle(\xi_\lambda)$, obtained by averaging q over the tori Λ_{ξ_λ} . Then we have, in C^1 -sense in $\xi_\lambda \in A_\lambda$,

$$\langle q \rangle(\xi_\lambda) = \langle q \rangle_{\Lambda_{\xi_\lambda}} \rightarrow \langle q \rangle_{\Lambda_\xi} = \langle q \rangle(\xi), \text{ as } \lambda \rightarrow 0.$$

Hence, noticing the properties of the good value G , for every $\xi_\lambda \in \Gamma_{a, \lambda}$ and λ small enough, we get

$$|d\langle q \rangle(\xi_\lambda)| \geq \frac{\alpha}{2}. \quad (3.56)$$

Notice that we have also the differentials of $p_\lambda(\xi_\lambda)$ and $\langle q \rangle(\xi_\lambda)$ in every $\xi_\lambda \in \Gamma_{a, \lambda} \cap A_{\alpha, d, \lambda}$ are \mathbb{R} -linearly independent:

$$\omega_\lambda(\xi_\lambda) \wedge d\langle q \rangle(\xi_\lambda) \neq 0. \quad (3.57)$$

Let us define the set of (new) *good values* for $P_{\varepsilon, \lambda}$,

$$\mathcal{G}_\lambda(\alpha, E, G) = \bigcup_{\xi_\lambda \in \Gamma_{a, \lambda} \cap A_{\alpha, d, \lambda}} \langle q \rangle(\xi_\lambda) = \bigcup \{ \langle q \rangle(\xi_\lambda) : \xi \in \Gamma_a \cap A_{\alpha, d} \}. \quad (3.58)$$

It is true that the measure of the complement of $\mathcal{G}_\lambda(\alpha, E, G)$ in $\bigcup_{\xi_\lambda \in \Gamma_{a, \lambda}} \langle q \rangle(\xi_\lambda)$ is small, is of order $\mathcal{O}(\alpha)$, when α is small and d is kept fixed.

If $K \in \mathcal{G}_\lambda(\alpha, E, G)$, then there exists a unique $\xi \in \Gamma_a \cap A_{\alpha, d}$ such that $\langle q \rangle(\xi_\lambda) = \langle q \rangle_{\Lambda_{\xi_\lambda}} = K$, and the corresponding KAM torus $\Lambda_{\xi_\lambda} \subset p_\lambda^{-1}(E)$ is still uniformly Diophantine H_{p_λ} -invariant Lagrangian torus. Moreover, the H_{p_λ} -flow on Λ_{ξ_λ} is quasi-periodic of the Diophantine constant frequency $\omega_\lambda(\xi_\lambda)$, satisfying (3.56) and (3.57). Therefore, these basis assumptions on the dynamic of H_{p_λ} allow us to carry out, microlocally near Λ_{ξ_λ} , the construction of Birkhoff normal form for $P_{\varepsilon, \lambda}$.

Remark 3.5. The principle of the Birkhoff normalization is to use excessively canonical transformations near a Diophantine invariant torus (like Λ_{ξ_λ}), to conjugate the operator $P_{\varepsilon, \lambda}$ to a new operator, whose total symbol is independent of angle variables x , and homogeneous polynomial to high order in (ξ, h, ε) . This conjugation, with the help of the Egorov Theorem [20], is by means of analytic Fourier integral operators [17]. The Diophantine condition of the torus (Λ_{ξ_λ}) is indispensable for this construction. We refer to [8], [29], and [28] for the Birkhoff normal form procedure.

So as a result of Ref. [8] (Section 7.3), one obtains asymptotic spectral results for the operator $P_{\varepsilon, \lambda}$, that are similar to ones of the operator P_ε in the integrable case.

Theorem 3.6 ([8]). *Let $P_{\varepsilon, \lambda}$ be the operator in (3.54), with leading term $p_{\varepsilon, \lambda}$ (3.55), satisfying all assumptions given in this section. We work in the regime $h \ll \varepsilon = \mathcal{O}(h^\delta)$, for $0 < \delta < 1$, and $\lambda > 0$ small enough. Let $a = (E, G)$ be any point in U such that $G \in \mathcal{G}(\alpha, d, E)$, with $\alpha > 0$ small enough, and $d > 0$ fixed, as in Definition 2.6. We assume that $\lambda \ll \alpha^2$, and define the set $\mathcal{G}_\lambda(\alpha, E, G)$ as in (3.58).*

For each $K \in \mathcal{G}_\lambda(\alpha, E, G)$, there exists a KAM torus $\Lambda_{\xi_\lambda} = p_\lambda^{-1}(E) \cap \langle q \rangle^{-1}(K)$ of H_{p_λ} -flow's frequency $\omega_\lambda(\xi_\lambda)$ as already discussed above, and a canonical transformation

$$\kappa_\infty = (x, \xi) : \text{neigh}(\Lambda_{\xi_\lambda}, T^*M) \rightarrow \text{neigh}(\xi = \xi_\lambda, T^*\mathbb{T}^2), \quad (3.59)$$

mapping Λ_{ξ_λ} to $\mathbb{T}^2 \times \{\xi_\lambda\}$, such that

$$p_\lambda \circ \kappa_\infty^{-1} = p_\lambda(x, \xi) = p_{\lambda, \infty}(\xi) + \mathcal{O}(\xi - \xi_\lambda)^\infty, \quad (3.60)$$

where $p_{\lambda, \infty}$ is a smooth function, depends on ξ only, and admits the Taylor expansion at ξ_λ of the form

$$p_{\lambda, \infty}(\xi) = E + \omega_\lambda(\xi_\lambda) \cdot (\xi - \xi_\lambda) + \mathcal{O}(\xi - \xi_\lambda)^2. \quad (3.61)$$

Let $\eta \in \mathbb{Z}^2$ be the Maslov indices (see Definition 2.8) and $S \in \mathbb{R}^2$ be the action integrals, defined as in (2.33) of the fundamental cycles of Λ_{ξ_λ} , suitable with the canonical transformation (3.59). Then all the eigenvalues μ of $P_{\varepsilon, \lambda}$ in the rectangle $R^{(E, K)}(\varepsilon, h)$ defined by (2.31) with K instead of G , are given as the image of a portion of $h\mathbb{Z}^2$, with modulo $\mathcal{O}(h^\infty)$, by a smooth function $P_\lambda(\xi, \varepsilon; h)$ of ξ in a neighborhood of $(\xi_\lambda, \mathbb{R}^2)$ and ε, h in neighborhoods of $(0, \mathbb{R})$:

$$\sigma(P_{\varepsilon, \lambda}) \cap R^{(E, K)}(\varepsilon, h) \ni \mu = P_\lambda\left(\xi_\lambda + h\left(k - \frac{\eta}{4}\right) - \frac{S}{2\pi}; \varepsilon, h\right) + \mathcal{O}(h^\infty), \quad k \in \mathbb{Z}^2. \quad (3.62)$$

Moreover, P_λ is real valued for $\varepsilon = 0$, admits an asymptotic expansion in (ξ, ε, h) of the form (2.35). In particular, the h -leading term of P_λ is of the form

$$p_{0,\lambda}(\xi, \varepsilon) = p_{\lambda,\infty}(\xi) + i\varepsilon\langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2), \quad (3.63)$$

where $p_{\lambda,\infty}(\xi)$ is given in (3.60), and $\langle q \rangle(\xi)$ is the expression of the average of q over the tori Λ_ξ close to Λ_{ξ_λ} , in the previous coordinates.

We notice that the canonical transformation κ_∞ in (3.59), which satisfies the properties (3.60) and (3.61), is locally valid for every KAM torus in a neighborhood of Λ_{ξ_λ} . Hence, the construction in the above theorem is locally valid for any given good value $a = (E, G) \in U$, and for $K \in \mathcal{G}_\lambda(\alpha, E, G)$.

We can see that the results in (3.62) and (3.63) are very similar to ones in (2.40) and (2.41) of Sec. 2.2. Hence the way to construct a spectral asymptotic pseudo-lattice here is the same as one of Sec. 2.2. Therefore, in order to avoid repetitions, we shall skip some technical details.

Eq. (3.62), with the aid of (3.63), provides a micro-chart, denoted by f_λ , of the spectrum of $P_{\varepsilon,\lambda}$ on the rectangle $R^{(E,K)}(\varepsilon, h)$. This micro-chart satisfies (2.44) and (2.45), with P replaced by P_λ and G by K . It is true that there exists locally a Cantor family of such micro-charts. Moreover, noticing the fact that the union of the invariant KAM tori is of full measure, thus this family is a pseudo-chart.

In the reduced form, the asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ of $\tilde{f}_\lambda := f_\lambda \circ \chi$, with χ given by (2.38), for the C^∞ -topology in a neighborhood of $(E, K) \in \mathbb{R}^2$, satisfies

$$\tilde{f}_\lambda := \tilde{f}_{\lambda 0} + \mathcal{O}\left(\varepsilon, \frac{h}{\varepsilon}\right),$$

uniformly for h, ε small and $h \ll \varepsilon$. Here $\tilde{f}_{\lambda 0}$ is the leading term of \tilde{f}_λ . Moreover, we have also a similar result as one in (2.47) that

$$\tilde{f}_{\lambda 0} = \tau_a + \psi^{-1}, \quad (3.64)$$

where

$$\psi(\xi) = (p_{\lambda,\infty}(\xi), \langle q \rangle(\xi)), \quad (3.65)$$

with ξ in a neighborhood of $(\xi_\lambda, \mathbb{R}^2)$, and τ_a is a locally constant in $a \in U$.

Thus we can state that the spectrum of $P_{\varepsilon,\lambda}$ on the domain $U(\varepsilon)$ satisfies all hypothesis of an asymptotic pseudo-lattice, according Definition 2.15. In conclusion, we have:

Theorem 3.7. $(\sigma(P_{\varepsilon,\lambda}), U(\varepsilon))$ is an asymptotic pseudo-lattice.

Therefore, we can define the spectral monodromy of the operator $P_{\varepsilon,\lambda}$ due to the monodromy of the asymptotic pseudo-lattice $(\sigma(P_{\varepsilon,\lambda}), U(\varepsilon))$, as discussed in Sec. 2.2 (see

also [35] for more detail). That is the class $[\mathcal{M}_{sp}] \in \check{H}^1(U, GL(2, \mathbb{Z}))$ in the Čech cohomology group, defined from the 1-cocycle of integer linear transition maps $\{M_{ij}\}_{i,j \in \mathcal{J}}$, given in (2.49). In the context of this section, we have

$$M_{ij} = d(\tilde{f}_{i\lambda,0} \circ (\tilde{f}_{j\lambda,0})^{-1}), \quad i, j \in \mathcal{J}. \quad (3.66)$$

Note that these M_{ij} don't depend on λ small enough, by the discreteness of the integer group $GL(2, \mathbb{Z})$. So the spectral monodromy of $P_{\varepsilon, \lambda}$ is independent of any small perturbation caused by λ . It is known that it is already independent of the classical parameters h, ε .

3.3 Discussions

The spectral monodromy is well defined for small non-selfadjoint perturbations of a selfadjoint classical operators in dimension 2, admitting a quasi-integrable unperturbed principal symbol, as discussed in Sec. 3.2. On the other hand, there exists a interesting result concerning a geometric invariant of quasi-integrable systems, given by Broer and co-worker, Ref. [10]. It showed the construction of a global Whitney smooth conjugacy (global KAM) between a quasi-integrable system and an integrable one. That means the global quasi-periodic stability of invariant Lagrangian tori in phase space, which allows them to define a monodromy for KAM tori. For simplicity, we say it as the KAM monodromy. Naturally, we discuss in this section the relationship between the spectral monodromy and the KAM monodromy.

We consider a H_p -invariant Lagrangian torus $\Lambda_a = F^{-1}(a) \subseteq X$, with $a \in U$. Let (V, κ) be a local angle-action chart near Λ_a , and Φ_λ be the corresponding local KAM conjugacy given by Theorem 3.2. We define on the phase space the Lagrangian torus

$$\Lambda_{a,\lambda} = \kappa^{-1} \circ \Phi_\lambda \circ \kappa(\Lambda_a) \subseteq X.$$

A interesting result from [10] showed that if the action coordinates $\xi_a \in A_{\alpha,d}^*$, where the set $A_{\alpha,d}^*$ is defined in Remark 3.3, then the corresponding KAM torus $\Lambda_{a,\lambda}$ in the phase space doesn't depend on the choice of such angle-action chart (V, κ) . In this way, we can obtain all invariant KAM tori on X . In fact, there exists a nowhere dense subset $C \subset U$ and the measure of the set $U \setminus C$ tends to 0 as λ tends to zero. Accordingly, the set of Liouville tori $F^{-1}(C) \subset X$ is also nowhere dense of full measure. Moreover, by gluing together local KAM conjugacies with the help of a partition of unity, one obtains a global conjugacy as a finite convex linear combination of local conjugacies. There exists so a C^∞ -diffeomorphism (the global conjugacy) Φ on the entire space X , whose restriction on $F^{-1}(C)$ conjugates each Λ_a , with $a \in C$, to the H_{p_λ} -invariant KAM torus $\Lambda_{a,\lambda}$. This conjugacy provides an equivalence (smooth in the sense of Whitney) between the integrable system and its small perturbation on a nowhere dense set of large measure.

Hence the union of all *global KAM tori* in X is the nowhere dense subset $\bigcup_{a \in C} \Lambda_{a,\lambda} \subseteq X$. Moreover, the measure of KAM tori in X is large,

$$\mu(X \setminus \bigcup_{a \in C} \Lambda_{a,\lambda}) = \mathcal{O}(\alpha)\mu(X).$$

Thus one obtains a Whitney smooth foliation of invariant KAM tori over a nowhere dense set with complement of small measure.

We notice here that the functions p_λ and the average function $\langle q \rangle = \langle q \rangle_{\Lambda_{a,\lambda}}$ are Whitney smooth, independent (in seeing (3.57)), and in involution when restricted to the nowhere dense set of large measure of the global KAM tori. One says that this is the aspect of *integrability for perturbed systems*. In the context $n = 2$, this result which is global, is stronger than one of Ref. [14], which proved locally the existence of a system with those properties. On the other hand, the previous system found in our work is different from one of the last paper, which is locally given by the components of the frequency $\omega_\lambda(\xi_\lambda)$ of the H_{p_λ} -flow on local KAM tori, expressed as functions on the phase space.

It is known that on the Liouville tori of integrable systems, the classical monodromy is well defined by Duistermaat [16]. Whether one can define such a geometric invariant for the KAM tori of quasi-integrable perturbed systems? The answer is positive if unperturbed integrable systems are globally non-degenerate, given in [10]. The KAM monodromy is defined as the obstruction against global triviality of a \mathbb{Z}^2 -bundle over the whole base U , denoted by \mathcal{F} , which is an extension from the \mathbb{Z}^2 -bundle $H_1(\Lambda_{a,\lambda}, \mathbb{Z}) \rightarrow a \in C$. Here $H_1(\Lambda_{a,\lambda}, \mathbb{Z})$ is the first homology group of $\Lambda_{a,\lambda}$.

Let ξ_λ be local action coordinates, given by (3.59), of the global KAM tori $\Lambda_\lambda \simeq \Lambda_{\xi_\lambda}$. We have $\psi(\xi_\lambda) = (p_\lambda(\xi_\lambda), \langle q \rangle(\xi_\lambda)) := b$, where the map ψ is defined by (3.65). So the map ψ^{-1} give the same local action variables of Λ_λ . It is classical that in a reverse manner of (2.33), there is a one-to-one correspondence between the compositions of $d\psi^{-1}(b)$ and periodic flows of period 1 on Λ_λ , which form a basis of $H_1(\Lambda_\lambda, \mathbb{Z})$, see [16].

On the other hand, we look at now the transition maps from the spectral monodromy $[\mathcal{M}_{sp}]$ of $P_{\varepsilon,\lambda}$. Using (3.64) and (3.66) we get

$$d(\psi_i^{-1} \circ \psi_j) = M_{ij} \in GL(2, \mathbb{Z}).$$

This result asserts that the above homology bundle \mathcal{F} has an integer linear structure group. The transition maps between local trivializations of the bundle are $\{{}^t M_{ij}^{-1}\}$. Therefore, the spectral monodromy recovers completely the KAM monodromy of quasi-integrable systems. Moreover, we have a similar result as one of the integrable case.

Theorem 3.8. *The spectral monodromy of $P_{\varepsilon,\lambda}$ is the adjoint of the KAM monodromy.*

In conclusion for this paper, the spectral monodromy is well defined directly from the spectrum of small non-selfadjoint perturbations of a selfadjoint operators admitting, either a completely integrable (semi-)classical principal symbol, or a quasi-integrable

one with the globally non-degenerate condition. In both cases, the spectral monodromy allows to recover, either the classical monodromy of the Liouville invariant tori, defined by Duistermaat [16], or the monodromy of the KAM invariant tori, defined by Broer and co-workers [10]. That illustrates clearly the principle idea of semiclassical analysis that classical mechanics is the semiclassical limit of quantum mechanics.

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