DEFINABLE TRIANGULATIONS WITH REGULARITY CONDITIONS

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Summary of the thesis

It has been known for more than 40 years - since papers of Whitney and Łojasiewicz - that analytic and semianalytic subsets of Euclidean spaces admit stratifications with Whitney regularity conditions, a result later generalized to subanalytic (Hironaka, Łojasiewicz, Stasica, Wachta) and finally to definable subsets (Ta Le Loi). Since Łojasiewicz's paper, it has also been known that semianalytic and subsequently, subanalytic (Hardt, Hironaka) and definable (van den Dries) sets are triangulable.

A challenging problem stated by Łojasiewicz and Thom was to combine the both results, i.e. to construct a triangulation of semi(sub)analytic sets which is a stratification with Whitney conditions. A main difficulty was that the construction of a Whitney stratification was by downward induction on dimension in contrast to the triangulation which goes by upward induction on dimension. It was not clear how to overcome this divergence.

A first positive solution to the problem was given by Masahiro Shiota in 2005. In his eight-page article *Whitney triangulations of semialgebraic sets* concerning semialgebraic case, he proposed a solution based on a technique of controlled tube systems developped in his book *Geometry of subanalytic and semialgebraic sets*. However, his proof is difficult to understand.

In the thesis *Definable triangulations with regularity conditions* I give a direct constructive solution to the problem based on the theory of weakly Lipschitz mappings developped in the paper *Invariance of regularity conditions under definable, locally Lipschitz, weakly bi-Lipschitz mappings* and on Guillaume Valette's description of Lipschitz structure of definable sets (paper *Lipschitz triangulations*). The solution is general in the sense that it concerns an arbitrary o-minimal structure on the ordered field of real numbers $\mathbb{R}$ and, moreover, in the sense that I describe a class $\mathcal{T}$ of regularity conditions including the Whitney and the Verdier conditions, such that for any condition $\mathcal{Q}$ from the class $\mathcal{T}$ a definable triangulation with the condition $\mathcal{Q}$ is possible.

Roughly speaking, the final result (Theorem 6.2.2) is derived from existence of a definable, locally Lipschitz, weakly bi-Lipschitz triangulation (Theorem 5.2.2), which in some sense (see Theorem 3.2.1) preserves regularity conditions. To construct such a triangulation, I first use Guillaume Valette's theorem about definable bi-Lipschitz homeomorphism, which reduces the general case to the one, to which the classical construction of triangulation can be applied.