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# The complex Monge-Ampère equation and its applications in geometry

PhD thesis

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## Contents

# 1. Introduction

Pluripotential theory on compact Kähler manifolds has become a separate branch of research in the last decade. Some results have been known earlier (especially in the case of projective spaces) yet it was very recently, mainly due to works of Guedj, Zeriahi and their collaborators ([GZ1], [GZ2], [BGZ], [CGZ], [EGZ], [BEGZ]) as well as Kołodziej ([K2], [K3]) when the theory has grown and separated from its flat counterpart (the pluripotential theory in domains in $\mathbb{C}^n$). It is worth mentioning that the main motivation for this development were the applications in geometry and complex dynamics.

Like any other new theory, many of its results are far from being optimal. Also many natural problems remain open. The methods are often borrowed from the flat theory, however the "new" theory is not just a generalization of the "old" one and not every "flat" theorem has its "Kähler" counterpart.

The aim of this work is to systematically describe these concepts, to explain the basic notions and ideas and to show their applications as well as to indicate possible directions of future studies. Of course the accent falls onto authors' own results taken from the papers [Di1], [Di2], [Di3], [Di4] and from the joint paper with Z. Zhang [DZ].

The autor would like to emphasize that analysis will be the main field of study in the thesis. Thus the geometric part is treated just as an explanation of the structure of the underlying spaces for the theory, or as a motivation for the analytic subtleties. As a consequence of this the geometric sections differ significantly from the others - majority of the notions defined have been only sketched usually without examples (for which the reader is directed to the cited literature).

The author's intention was to put his own results in a perspective of the yet-grown theory. Due to length limitation some important results have been only sketched. Other developments, in turn, were described in a much more complete manner than it is usually done. In such cases the author wanted to emphasize some of the important tools that were used in his original results.

The manuscript is divided into four sections. The first one contains the basic pluripotential theory in domains in $\mathbb{C}^n$ (i.e. the flat theory). The first subsection is devoted to the notion of currents. Author's contribution here is the explicit example of $(2, 2)$-forms in $\mathbb{C}^4$ whose wedge product is not positive (this is taken from [Di4]). Next we sketch the notion of pluri-fine topology, followed by the classical pluripotential theory (including the Monge-Ampère operator in the case of bounded plurisubharmonic functions, capacities and extremal functions). The next topic is the theory of Cegrell classes. This section finishes with the presentation of one of the author's main results: an inequality for mixed Monge-Ampère measures, taken from [Di2].

In the second section all the geometric notions and concepts used in the thesis have been collected. This part starts by introducing the compact Kähler manifolds. Some examples are given. Next we sketch the main concepts in the theory of divisors and (line) bundles. The section ends with a discussion of special divisors (ample, big and nef ones) and with the notion of canonical bundles and Chern classes.

Third section is devoted to the Monge-Ampère equation on compact Kähler manifold. This is the heart of the whole thesis containing in particular most of the author's original results. After defining the needed notions we consider the geometric interpretation of the solutions of the mentioned equation and we sketch the proof of the Calabi-Yau theorem as well as its generalization. Next we discuss an important generalization of Kähler forms- the so called big forms. We show the continuity of the solutions in a particular case (this is a result of Z. Zhang, however the full proof has been taken from [DZ]). In the next subsections we discuss the Monge-Ampère equation in Cegrell classes. We subsequently sketch the proof of existence of solutions (the Guedj-Zeriahi theorem) and a proof of uniqueness (this is an author's result taken from [Di3]). Next we discuss the problem of stability of the solutions. Three important theorems are given, due to Błocki [Bl5], Eyssidieux, Guedj and Zeriahi [EGZ] and Kołodziej [K3]. The last one is generalized and strenghtened (the argument is taken from [DZ]). The last subsection is devoted to the higher regularity of the solutions of the Monge-Ampère equation.

The fourth section describes the Kähler-Ricci flow, its applications in geometry and some problems concerned with this flow which can be attacked with pluripotential methods. This section may be regarded as an ilustration of current research trends in the field and as an overview of some of the open problems.

We wish to point out that the continuity of the solutions of the Monge-Ampère equation in the case of big forms was recently used by Song and Tian [ST2] and Tosatti [To] in some geometric problems. The inequality for mixed Monge-Ampère measures from [Di2] and the methods of the proof of uniquenses from [Di3] were, in turn, used by Boucksom, Eyssidieux, Guedj and Zeriahi in [BEGZ].

We would also like to emphasize that pluripotential theory has become recently a very efficient tool in complex dynamics. Especially problems related to regularity of potentials for Green currents or special type measures seem to be strongly linked with the regularity theory for the complex Monge-Ampère equation discussed in the third section. We shall not discuss these connections further in the thesis, however we refer to a recent paper of Dinh, Nguyen and Sibony [DNS] where some results supporting these expectations may be found.

**Notation and conventions.** In the thesis we shall use the standard notations in analysis. When estimating some terms we shall often denote the constants by $C$. In case when there are many such constants in order to avoid confusion we shall enumerate them. The Lebesgue measure is denoted by $d\lambda$ and we shall keep this notation also in the manifold case. Consistently with the historical development of mathematics 0 is **not** a natural number.

In some cases there is no fixed terminology for the considered notions. Thus apart of the chosen by the author name for such a notion we shall list also the other alternatives existing in the literature. We shall proceed similarly also if the notation is not fixed yet (like in the case of Cegrell classes on manifolds).

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During my studies I had the opportunity to visit the University of Toulouse (France) due to an invitation from professor Ahmed Zeriahi, as well as the University of Umea (Sweden) due to an invitation from professor Urban Cegrell. These visits enriched my mathematiacal (and not only mathematical) viewpoint and contributed a lot to the creation of this thesis. Thus I wish to thank both professors and both Universities for the hospitality during my stays.

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## 2. Basic notions and definitions

### 2.1. Pluripotential theory.

2.1.1. *Currents.* Currents are basic tools in pluripotential theory. They can be thought as a generalization of differential forms. Herebelow we start the discussion of currents from the very beginning:

**Definition 2.1.1 (Current).** *Let $\Omega$ be a domain in $\mathbb{R}^n$. A current of degree $p$ in $\Omega$ is a differential $p$-form defined in $\Omega$ whose coefficients are distributions. The space of all $p$-currents in $\Omega$ will be denoted by $\mathcal{D}_p(\Omega)$.*

**Remark 2.1.2.** *It is straightforward to generalize this notion to the setting of smooth manifolds.*

A $p$-current acts on test forms of degree $(n-p)$ (i.e. $(n-p)$-forms whose coeffients are $\mathcal{C}_0^\infty$ functions) in the following natural way:

Let

$$\phi = \sum_{|J|=n-p}{}' \phi_J dx_J$$

be a test form (here $J = (j_1, j_2, \cdots, j_{n-p})$ is a $(n-p)$-tuple of indices $dx_J := dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_{n-p}}$ and the $'$ sign denotes summation over ordered multi-indices i.e. we assume $1 \leq j_1 < \cdots < j_{n-p} \leq n$). If $\Theta$ is a current of degree $p$ one can formally write $\Theta$ as

$$\Theta = \sum_{|I|=p}{}' \Theta_I dx_I$$

with $\Theta_I$ being distributions. The action of $\Theta$ on $\phi$ is defined by

$$\Theta(\phi) := \sum_{|J|=n-p}{}' \sum_{|I|=p}{}' \Theta_I(\phi_J) dx_I \wedge dx_J.$$

It is obvious that only terms for which $I$ is complementary to $J$ (i.e $I \cup J = \{1, 2, \cdots, n\}$) contribute to the sum.

It the sequel we shall often define currents simply dy defining their actions on the space of test forms of corresponding degree.

**Remark 2.1.3.** *All algebraic operations such as summation or multiplication by a function which are meaningful for distributions can be naturally defined in the setting of currents.*

A basic example is the *current of integration*:

**Example 2.1.4.** *Suppose $X$ is a hypersurface in $\Omega$. If $\phi$ is a $(n-1)$-form with $\mathcal{C}_0^\infty(\Omega)$-coefficients one defines the current of integration $[X]$ by*

$$[X](\phi) = \int_X \phi.$$

*For example, if $X$ is given by $\{ x = (x_1, \cdots, x_n) \in \Omega |\ x_n = 0 \}$ then $[X]$ is equal to $\delta_{(x',0)} dx_n$, where $\delta_{(x',0)}$ is the valuation distribution*

$$\delta_{(x',0)}(\psi) := \psi(x', 0).$$

Consider now the complex setting. Since $dz_i$ and $d\bar{z}_j$ constitute a basis for the space of 1-forms there is a natural splitting for $r$-forms in $\Omega \subset \mathbb{C}^n$

$$\mathcal{C}_p(\Omega) = \bigoplus_{p+q=r} \mathcal{C}_{p,q}(\Omega),$$

where each $\mathcal{C}_{p,q}(\Omega)$ consists of forms of type

$$\phi = \sum_{|I|=p,\ |J|=q} {}' \phi_{I\bar{J}} dz_I \wedge d\bar{z}_J,$$

(here, as before, $dz_I := dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}_{\bar{J}} := d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ and both multiindices are ordered).

This naturally leads to a definition on complex $(p, q)$-currents:

**Definition 2.1.5.** *A $(p, q)$-complex current is a $(p, q)$-form with complex distributions as coefficients. The space of complex $(p, q)$-currents will be denoted by $\mathcal{D}_{(p,q)}(\Omega)$.*

**Remark 2.1.6.** *Again the notion can be defined in a complex manifold setting in the obvious way.*

As an example we consider the current of integration $[Z] = [Z_n]$ over a complex hypersurface $Z = \{ z = (z_1, \cdots, z_n) \in \Omega |\ z_n = 0 \}$. This is a $(1, 1)$-current equal to $i\delta_{(z',0)} dz_n \wedge d\bar{z}_n$.

From now on throughout the thesis we will consider only complex currents. Therefore we shall call them simply currents.

**Definition 2.1.7 (Real current).** *A current is called real if $\Theta = \bar{\Theta}$.*

An obvious necessary condition for $(p, q)$-current to be real is $p = q$. In this case the necessary and sufficient condition for a current $\Theta$ to be real is that its coefficients $\Theta_{I\bar{J}}$ satisfy (as complex distributions) equality

$$\bar{\Theta}_{I\bar{J}} = \Theta_{J\bar{I}}.$$

In particular, coefficients $\Theta_{I\bar{I}}$ are real distributions.

From now on we will be interested only in currents of type $(p, p)$ and the main focus will be on real currents.

**Example 2.1.8.** *$[Z_n]$ is an example of a real $(1, 1)$-current. It can be proved that the current of integration along any complex submanifold of dimension p is a real current of type $(n - p, n - p)$.*

Another special class of currents heavily used in pluripotential theory are the *positive currents.*

**Definition 2.1.9 (Positive current).** *A real current $T$ of bidegree $(k, k)$ is called positive if for any $(1, 0)$-test forms $\alpha_1, \cdots, \alpha_{n-k}$ we have*

$$T \wedge i\alpha_1 \wedge \overline{\alpha_1} \wedge i\alpha_2 \wedge \overline{\alpha_2} \wedge \cdots \wedge i\alpha_{n-k} \wedge \overline{\alpha_{n-k}} \geq 0.$$

In [LG] the following result is proved (it shows in particular that positive currents are very special class of complex currents):

**Proposition 2.1.10.** *Coefficients of positive currents are complex measures (i.e. distributions of order 0).*

Herebelow we give two basic examples of positive currents:

**Example 2.1.11.** *The current $[Z_n]$ and, more generally, the current of integration over arbitrary complex submanifold is a positive current.*

The next example shows the fundamental link between plurisubharmonic functions and positive currents:

**Example 2.1.12.** *If $u$ is a psh function then $i\partial\bar{\partial}u$ is a positive $(1,1)$-current. Here and below $\partial$ and $\bar{\partial}$ are the standard operators representing respectively the $(1,0)$ and $(0,1)$ part of the exterior differentiation operator $d$. The positivity of $i\partial\bar{\partial}u$ can be seen as follows: if we rearrange terms in the wedge product*

$$i\partial\bar{\partial}u \wedge i\alpha_1 \wedge \overline{\alpha_1} \wedge i\alpha_2 \wedge \overline{\alpha_2} \wedge \cdots \wedge i\alpha_{n-1} \wedge \overline{\alpha_{n-1}}$$

*we shall end up simply with the Levi form of $u$ evaluated at some vector dependent on $\alpha_1, \cdots, \alpha_{n-1}$.*

**Remark 2.1.13.** *The example above can be, in a way, reversed. Namely, if $T$ is a positive $(1,1)$-current then it was proved in* [LG] *that locally one can find a plurisubharmonic function $u$ such that $i\partial\bar{\partial}u = T$.*

A very important operation in pluripotential theory is the wedge product of currents. In general one cannot perform it, since one cannot multiply the coefficients which are distributions. However, in some special cases one can define the product. If, for example, at least one of the currents involved has smooth coefficients then the operation is well defined.

A basic question arises if the wedge product of positive currents is still positive. A crucial fact in this direction is the following theorem:

**Theorem 2.1.14.** *Let $\Theta$ be a positive $(p,p)$-current and $T$ be a positive $(1,1)$-current. Assume, for simplicity, that one of these currents has smooth coefficients. Then the wedge product $\Theta \wedge T$ is a positive $(p+1,p+1)$-current.*

**Remark 2.1.15.** *In the theorem above it is important that one of the currents is of type $(1,1)$. Bedford and Taylor* [BT1] *and, independently, Harvey and Knapp* [HK] *have shown that for currents of higher bidegree this theorem is not true. The example below (taken from* [Di4]*) shows that the wedge product of two smooth positive $(2,2)$-currents in $\mathbb{C}^4$ may fail to be positive.*

Before we give the example we need an auxiliary proposition:

**Proposition 2.1.16.** *A current with smooth coefficients $\alpha = \sum'_{|I|=2,|J|=2} a_{IJ}\, e_I \wedge \overline{e_J}$ is positive if and only if for all $s \in M$ $\;sA\bar{s}^T \geq 0$. Here $e_I = dz_{i_1} \wedge dz_{i_2}$, $I = (i_1,\ i_2)$, $A$ is the matrix $[a_{IJ}]_{I,J}$, $s^T$ means transposed (i.e. column) vector and $M$ is the complex cone defined by*

$$M := \{(s_1, s_2, s_3, s_4, s_5, s_6) \in \mathbb{C}^6 |\ s_1s_6 + s_3s_4 = s_2s_5\}.$$

*(So, $A$ is in a sense positive definite when restricted to vectors from $M$).*

*Proof.* Take arbitrary $(1,0)$-forms $\gamma = \sum_{j=1}^4 c_j dz_j$ and $\beta = \sum_{j=1}^4 b_j dz_j$. We must check that the coefficient in $\alpha \wedge i\gamma \wedge \overline{\gamma} \wedge i\beta \wedge \overline{\beta}$ is positive. But this is equivalent to $\alpha \wedge \gamma \wedge \beta \wedge \overline{\gamma \wedge \beta} \geq 0$, which after elementary operations leads to

$$\Theta(c_1, c_2, c_3, c_4, b_1, b_2, b_3, b_4) A \overline{\Theta(c_1, c_2, c_3, c_4, b_1, b_2, b_3, b_4)}^T \geq 0,$$

where $\Theta$ is the mapping

$$\Theta : \mathbb{C}^8 \ni (z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4) \longmapsto (z_1 w_2 - z_2 w_1, -z_1 w_3 + z_3 w_1, \\ , z_1 w_4 - z_4 w_1, z_2 w_3 - z_3 w_2, -z_2 w_4 + z_4 w_2, z_3 w_4 - z_4 w_3) \in \mathbb{C}^6.$$

Now the claimed result follows from the (rather surprising) observation that $\Theta(\mathbb{C}^8) = M$. □

Now we can write down the explicit example:

**Example 2.1.17.**

$$\zeta = \frac{19}{4} dz_1 \wedge dz_2 \wedge \overline{dz_1 \wedge dz_2} + \frac{19}{4} dz_3 \wedge dz_4 \wedge \overline{dz_3 \wedge dz_4} + dz_1 \wedge dz_3 \wedge \overline{dz_1 \wedge dz_3} + \\ + dz_1 \wedge dz_4 \wedge \overline{dz_1 \wedge dz_4} + dz_2 \wedge dz_3 \wedge \overline{dz_2 \wedge dz_3} + dz_2 \wedge dz_4 \wedge \overline{dz_2 \wedge dz_4} - \\ - \frac{21}{4} dz_1 \wedge dz_2 \wedge \overline{dz_3 \wedge dz_4} - \frac{21}{4} dz_3 \wedge dz_4 \wedge \overline{dz_1 \wedge dz_2},$$

$$\eta = \frac{19}{4} dz_1 \wedge dz_2 \wedge \overline{dz_1 \wedge dz_2} + \frac{19}{4} dz_3 \wedge dz_4 \wedge \overline{dz_3 \wedge dz_4} + dz_1 \wedge dz_3 \wedge \overline{dz_1 \wedge dz_3} + \\ + dz_1 \wedge dz_4 \wedge \overline{dz_1 \wedge dz_4} + dz_2 \wedge dz_3 \wedge \overline{dz_2 \wedge dz_3} + dz_2 \wedge dz_4 \wedge \overline{dz_2 \wedge dz_4} + \\ + \frac{21}{4} dz_1 \wedge dz_2 \wedge \overline{dz_3 \wedge dz_4} + \frac{21}{4} dz_3 \wedge dz_4 \wedge \overline{dz_1 \wedge dz_2}$$

*Then*

$$\zeta \wedge \eta = (2(\frac{19}{4})^2 + 4 - 2(\frac{21}{4})^2) dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge \overline{dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4} = \\ = -6 dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge \overline{dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4},$$

*so* $\zeta \wedge \eta$ *is not positive.*

*Now let us prove that* $\zeta$ *is positive (*$\eta$ *goes the same way):*

*By our criterion and elementary calculations it boils down to checking that*

$$|z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + 5|z_6 - z_1|^2 - \frac{1}{4}|z_1 + z_6|^2 \geq 0$$

*for all* $z \in M$.

*But*

$$|z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + 5|z_6 - z_1|^2 - \frac{1}{4}|z_1 + z_6|^2 \geq |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + \\ + |z_6 - z_1|^2 - \frac{1}{4}|z_1 + z_6|^2.$$

*Since this is a homogenous expression, it is no restriction to assume $||z|| = 1$ (and still $z \in M$). But then*

$$|z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + |z_6 - z_1|^2 - \frac{1}{4}|z_1 + z_6|^2 \geq$$
$$\geq 1 - 2\Re z_1 \overline{z_6} - \frac{1}{4}|z_1 + z_6|^2 \geq 1 - 2|z_1 z_6| - \frac{1}{4}|z_1 + z_6|^2 =$$
$$= 1 - 2|z_2 z_5 - z_4 z_3| - \frac{1}{4}|z_1 + z_6|^2 \geq 1 - |z_2|^2 - |z_3|^2 - |z_4|^2 - |z_5|^2 - \frac{1}{4}|z_1 + z_6|^2 =$$
$$= |z_1|^2 + |z_6|^2 - \frac{1}{4}|z_1 + z_6|^2 \geq \frac{1}{4}|z_1 + z_6|^2 \geq 0.$$

For more informations regarding currents we refer to [Ho], [Kli], [LG] and [K4].

2.1.2. *The pluri-fine topology.* Since plurisubharmonic functions are not continuous in general it appears that the standard topology in Euclidean spaces is not well adjusted to this function class. Therefore one has to define a more subtle topology that takes into account these discontinuities. Thus, roughly speaking, the pluri-fine topology is the coarsest topology making all plurisubharmonic functions continuous. Herebelow we give a formal definition:

**Definition 2.1.18 (The pluri-fine topology).** *The topology in $\mathbb{C}^n$ defined by the basis*

$$\mathcal{U}_{B(z,r),\phi,a} = \{\, w \in B(z,r)|\ \phi(z) > a,\ \phi \in PSH,\ a \in \mathbb{R} \,\},$$

*where $B(z,r)$ is the ball with radius $r$ centered at $z$, is called pluri-fine topology.*

**Remark 2.1.19.** *Usually this topology is defined in a different way and then the definition above follows as a corollary. For more detailed study of this concept we refer to* [Kli].

**Observation 2.1.20.** *Any open set in the Euclidean topology is also open in the pluri-fine topology.*

To see how far is the new topology from the standard one we consider the following example:

**Example 2.1.21.** *It is known (see, for example,* [Kli]*) that there exists a plurisubharmonic function (in the unit ball, to fix ideas) $h \neq -\infty$ such that $\{\, z|\ h(z) = -\infty \,\}$ is a dense set. So, $\mathcal{U} = \{\, z|\ h(z) > -c \,\}$ is a pluri-fine open set (non empty for big enough $c > 0$), while $\mathcal{U}$ has empty Euclidean interior.*

**Observation 2.1.22.** *Since topology on manifolds is induced by the topology in local charts, we can also define the notion of pluri-fine topology on complex manifolds.*

To end up our short discussion concerning this notion we prove a crucial fact that we shall use later on:

**Theorem 2.1.23.** *Any non empty pluri-fine open set has positive Lebesgue measure.*

*Proof.* Of course it is enough to prove the claim for any basis set $\mathcal{U}_{B(z,r),\phi,a}$. Suppose on contrary that the Lebesuge measure of some set $\mathcal{U}_{B(z,r),\phi,a}$ is zero. Since this set is assumed to be non empty there exists $w \in B(z,r)$, such that $\phi(w) > a$. Fix a small radius $r'$ so that $B(w,r')$ is relatively compact in $B(z,r)$. Therefore, by upper semicontinuity of $\phi$ we obtain that $sup_{B(w,r')}\phi = c < +\infty$. But then by the mean value inequality for plurisubharmonic functions we obtain ( with $\lambda(A)$- the volume of the set $A$)

$$\begin{aligned} a < \phi(w) &\leq \int_{B(w,r')} \phi(x)d\lambda x/\lambda(B(w,r')) \leq \\ &\leq [a\lambda(B(w,r') \setminus \mathcal{U}_{B(z,r),\phi,a})]/\lambda(B(w,r'))+ \\ &+ [c\lambda(B(w,r') \cap U_{B(z,r),\phi,a})]/\lambda(B(w,r') \leq \\ &\leq a + c0 = a, \end{aligned}$$

a contradiction. □

2.1.3. *The Monge-Ampère operator for bounded plurisubharmonic functions.* The Laplace operator can be regarded as the trace of the complex Hessian of a function. Of course, if $n > 1$ the trace does not capture all the information the matrix contains. One needs in particular all the coefficients of the characteristic polynomial in order to determine the eigenvalues. This leads naturally to the study of the determinant (and also of all the intermediate minors) of the complex Hessian. This leads to the definition of the complex Monge-Ampère operator:

**Definition 2.1.24 (The complex Monge-Ampère operator).** *Given a $C^2$ smooth function n a domain in $\mathbb{C}^n$ the Monge-Ampère operator is defined by*

$$MA(u) := 4^n n! \det(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}).$$

In the language of differential forms, which is better adapted to analysis (and therefore shall be used throughout) one can write this operator (modulo a constant) as

$$MA(u) := 4^n n! \det(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k})d\lambda = (dd^c u)^n,$$

where, as usual, $d = \partial + \overline{\partial}$ and $d^c := i(\overline{\partial} - \partial)$, so $dd^c = 2i\partial\overline{\partial}$, and

$$(dd^c u)^n := \underbrace{dd^c u \wedge \cdots \wedge dd^c u}_{n-times}.$$

The Laplace operator, due to its linearity, can be extended to operate also on non-smooth functions (by means of distribution theory). This is how the potential theory has been built. In the case of plurisubharmonic functions one can of course always define $dd^c u$ as a positive $(1,1)$-current but defining higher order wedge products is problematic.

It turns out, however, that using the theory of currents one can define Monge-Ampère operators (as nonnegative Radon measures) for arbitrary locally bounded plurisubharmonic functions. More precisely the following is true:

**Theorem 2.1.25. ( Bedford-Taylor** [BT2],[BT3])
*Let $u_0, u_1, \cdots, u_n \in PSH \cap L^\infty_{loc}$. Then:*

- *for every $k \in \{1, \cdots, n\}$ $\quad u_0 dd^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_k$ is a well defined current. In particular one can define inductively* $dd^c u_0 \wedge dd^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_k := dd^c(u_0 dd^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_k)$,
- *$(dd^c u_0)^n$ is a nonnegative Radon measure;*
- *The operator is continuous for monotone (i.e. decreasing or increasing) sequences: $\{u_j^k\} \searrow u_j$, or $\{u_j^k\} \nearrow u_j$, $(u_j^k \in PSH \cap L^\infty)$ then*

$$u_0^k dd^c u_1^k \wedge dd^c u_2^k \wedge \cdots \wedge dd^c u_l^k \to u_0 dd^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_k,$$

*where the convergence is in the weak-* topology of currents.*

This shows that for any locally bounded plurisubharmonic function $u$ one can well define the Monge-Ampère operator acting on it yielding a nonnegative Borel measure associated to the function $u$. Of course the definition is coherent with the standard one if we consider smooth psh functions.

In the theorem above, a convergence with respect to monotone sequences is obtained. For Laplace operator, in turn, weak convergence of subharmonic functions implies weak convergence of their Laplacians. Hence it is natural to ask whether also the Monge-Ampère operator is continuous with respect to weak topology in the class of plurisubharmonic functions. Note that by classical properties of plurisubharmonic functions (see, for example [Ho]) weak convergence in $PSH$ is equivalent to convergence in $L^p_{loc}$ for any $p > 1$. An example due Cegrell shows, however, that the Monge-Ampère operator is discontinuous with respect to the weak topology:

**Example 2.1.26** ([Ce1], see also [CK1]). *There exists a sequence $u_j$ of locally bounded plurisubharmonic functions in $\mathbb{C}^2$ converging weakly (hence also in $L^p_{loc}$, $\forall p > 1$) towards a locally bounded plurisubharmonic function $u$ such that $(dd^c u_j)^n \to \mu \neq (dd^c u)^n$ for some measure $\mu$. Furthermore in the unit ball there exist a sequence $u_j$ and $u$ with the same property, such that additionally all the boundary values of $u_j$ and $u$ are the same.*

Below we state an important result, due to Bedford and Taylor (see [BT4]).

**Theorem 2.1.27.** *Let $u,\ v$ be locally bounded plurisubharmonic functions in a domain in $\mathbb{C}^n$. Then*

$$\chi_{\{u>v\}}(dd^c u)^n = \chi_{\{u>v\}}(dd^c \max(u,\ v))^n.$$

*($\chi_A$ is the characteristic function of the set $A$).*

Of course, the delicate point is that $u$ and $v$ need not be continuous (so the Borel plurifine open set $\{ u > v \}$ need not be open in the Euclidean topology). As a corollary one abtains a relatively easy proof of the following result, known as the comparison principle:

**Theorem 2.1.28** (**Comparison principle** [BT2], [BT3]). *Let $\Omega \subset \mathbb{C}^n$ be a domain and $u,\ v \in PSH(\Omega)$. Suppose the set $\{ u > v \}$ is relatively compact in $\Omega$. Then*

$$\int_{\{u>v\}} (dd^c u)^n \leq \int_{\{u>v\}} (dd^c v)^n.$$

The comparison priciple is arguably the most important tool in pluripotential theory. Several versions of it will appear throughout the thesis in different situations.

*Proof.* We follow an idea from [GZ2]. First, since the set $\{ u > v \}$ is relatively compact, we can modify $u$ and $v$ near the boundary, (which will not affect the quantities in the statement) so that $u = v$ near $\partial\Omega$. But then, by Stokes theorem (see [Bl1] for a rigorous justification) we have

$$\int_\Omega (dd^c u)^n = \int_\Omega (dd^c v)^n = \int_\Omega (dd^c max(u,\ v))^n.$$

Thus we get

$$\int_{\{u>v\}} (dd^c u)^n = \int_{\{u>v\}} (dd^c max(u,\ v))^n = \int_\Omega (dd^c max(u,\ v))^n -$$

$$- \int_{\{u\leq v\}} (dd^c max(u,\ v))^n \leq \int_\Omega (dd^c v)^n - \int_{\{u<v\}} (dd^c max(u,\ v))^n =$$

$$= \int_\Omega (dd^c v)^n - \int_{\{u<v\}} (dd^c v)^n = \int_{\{u\geq v\}} (dd^c v)^n.$$

Now exchanging $u$ with $u-\epsilon, \epsilon > 0$ we again obtain the relative compactness of the sets in question, so

$$\int_{\{u>v\}}(dd^c u)^n = lim_{\epsilon\to 0}\int_{\{u-\epsilon>v\}}(dd^c u)^n \leq lim_{\epsilon\to 0}\int_{\{u-\epsilon\geq v\}}(dd^c v)^n = \int_{\{u>v\}}(dd^c v)^n.$$

□

2.1.4. *Capacities.* As we have seen the Monge-Ampère operator is badly discontinuous with respect to convergence in $L^1$ topology. Moreover even in the one dimensional case some sets of Lebesgue measure zero behave (from potential point of view) as if they were big sets: for example the Newtonian potential is well defined for sets like a segment or arc in $\mathbb{C}$.

All this serves as evidence that measure theory cannot capture all the information encoded in the plurisubharmonic functions. One needs a theory that handles with a priori much smaller sets.

A satisfactory model was already in use in the case of the potential theory in the plane - the notion of a (Newtonian) capacity. However it relied heavily on the linear structure of the Laplace operator. Therefore a non-linear counterpart in higher dimensions was needed. This was accomplished by Bedford and Taylor in [BT3], where the *relative capacity* was introduced.

Herebelow we discuss the basic notions of capacities in complex analysis:

**Definition 2.1.29 (Relative capacity).** *Let $K\subset\Omega$ be a compact subset of an open set $\Omega\subset\mathbb{C}^n$. The relative capacity is defined as*

$$cap(K,\Omega) := \sup\{\int_K (dd^c u)^n |\ u\in PSH(\Omega),\ 0<u<1\}.$$

This definition can be extended to any Borel subset $E$ of $\Omega$ by defining

$$cap(E,\Omega) = cap_*(E,\Omega) := \sup\{cap(K,\Omega)\ |\ K\subset E,\ K compact\}.$$

**Remark 2.1.30.** *Since its introduction in* [BT3] *the relative capacity turned out to be a very efficient tool in pluripotential theory, behaving in many situations similarly to the Newtonian capacity in the planar case.*

The explicit computation of this capacity, except in a very special situations, is virtually impossible. Nevertheless one may introduce a function that, roughly speaking, realizes the supremum in the definition:

**Definition 2.1.31 (Relative plurisubharmonic extremal function).** *Let $K\subset\Omega$ be a compact subset of an open set $\Omega\subset\mathbb{C}^n$. Define the relative plurisubharmonic extremal function as*

$$u_{K,\Omega} := \sup\{v(z)\ |\ v\in\ PSH(\Omega),\ u\leq 0,\ u|_K\leq -1\}.$$

The upper semicontinuous regularization of $u$

$$u^*_{K,\Omega}(z) := limsup_{z'\to z}u_{K,\Omega}(z')$$

is a psh function.

Before we state the basic results concerning the connections between capacity and extremal functions we need to define the *hyperconvex* domains in $\mathbb{C}^n$.

**Definition 2.1.32 (Hyperconvex domain).** *A domain $\Omega\subset\mathbb{C}^n$ is called hyperconvex if there exists continuous negative plurisubharmonic exhaustion function $u$ - a function, such that*

$$\forall c<0\ :\ \ \{u\leq c\}\Subset\Omega.$$

**Remark 2.1.33.** *All hyperconvex domains are pseudoconvex but not vice versa. One might consider hyperconvex domains as multidimensional analogues of domains regular with respect to Laplace operator from the planar case (hovewer domain regular with respect to Laplace operator in* $\mathbb{C}^n$ *need not be hyperconvex as the example of Hartogs triangle shows).*

Below we list the main results concerning the above notions (see, for example [Kli]):

**Theorem 2.1.34.** *Let* $\Omega \subset \mathbb{C}^n$ *and* $K$ *be a compact subset in* $\Omega$*. Then*

(1) $K$ *is pluripolar if and only if* $cap(K, \Omega) = 0$*.*
(2) *The measure* $(dd^c\, u^*_{K,\Omega})^n$ *is supported in* $K$*.*
(3) $u^*_{K,\Omega} = -1$ *in the interior of* $K$ *and on* $\partial K$ *except possibly a pluripolar set.*
(4) *If* $\Omega$ *is additionally hyperconvex then*

$$cap(K, \Omega) = \int_K (dd^c\, u^*_{K,\Omega})^n = \int_\Omega -u^*_{K,\Omega}(dd^c\, u^*_{K,\Omega})^n.$$

Now we introduce the notion of convergence with respect to (relative) capacity:

**Definition 2.1.35 (Convergence with respect to capacity).** *A sequence of psh functions* $u_j$ *defined on an open set* $\Omega$ *is said to converge with respect to capcity to a psh function* $u$ *if for any compact subset* $K \subset \Omega$ *and any* $\epsilon > 0$ *we have*

$$lim_{j\to\infty} cap(K \cap \{\, |u_j - u| > \epsilon, \Omega \,\}) = 0.$$

The importance of this notion is due to the following result ([X1]):

**Theorem 2.1.36 (Xing's theorem).** *Let* $\{\, u_j^{(k)} \,\}_{j=1}^{\infty}$*,* $k = 1, \cdots, n$ *be sequences of locally uniformly bounded psh functions on some domainn* $\Omega$*. Suppose that these sequences converge with respect to capacity to the functions* $u^{(k)}$*. Then*

$$dd^c\, u_j^{(1)} \wedge \cdots \wedge dd^c\, u_j^{(n)} \to dd^c\, u^{(1)} \wedge \cdots \wedge dd^c\, u^{(n)},$$

*where the convergence is in the weak star topology. So, the Monge-Ampère operator is continuous with respect to convergence in capacity.*

Below we introduce yet another capacity, called *the Siciak capcacity.* While it is not straightforwardly related with the Monge-Ampère operator, its definition uses estimates on (special class of) plurisubharmonic functions. Quite often the connections between those two capacities serve as a technical tool in pluripotential theory, so one can exploit the maximum of the developed theory. We need several intermediate definitions before we discuss the new notion:

First of all, we define a special subset of psh functions defined on the whole $\mathbb{C}^n$. It is called *the Lelong class*:

**Definition 2.1.37 (Lelong class).** *The Lelong class of plurisubharmonic functions is defined by*

$$\mathcal{L}(\mathbb{C}^n) := \{u \in PSH(\mathbb{C}^n)|\ \limsup_{z\to\infty}(u(z) - log(1 + |z|)) \leq C_u < \infty\},$$

*where the constant* $C_u$ *depends only on* $u$*.*

These functions are also known as psh functions with logarithmic growth.

With this class one associates a special extremal function known as *Siciak-Zahariuta global extremal function* (below we shall call it the global extremal function for simplicity):

**Definition 2.1.38 (Global extremal function,** [S1], [Za]**).** *Let $K$ be a relatively compact subset in $\mathbb{C}^n$. Define*

$$V_K(z) := sup\{u(z)|\ u \in PSH(\mathbb{C}^n) \cap \mathcal{L}(\mathbb{C}^n),\ u|_K \le 0\}.$$

This function was introduced (in a different way) by Siciak in [S1]. The definition above is due to Zahariuta [Za].

It follows from the definition that $V_K$ is lower semicontinuous. By upper-regularization

$$V_K^*(z) := \limsup_{\zeta \to z} V_K(\zeta)$$

we obtain an upper semicontinuous function. It can be proved that $V_K^*$ is in fact plurisubharmonic, unless it is identically $+\infty$.

It is important to know when exactly $V_K^*$ is finite. A classical result in pluripotential theory allows classification of such sets ([S2]):

**Theorem 2.1.39.** *Let $K$ be relatively compact subset in $\mathbb{C}^n$. Then the following conditions are equivalent:*

(1) *$K$ is not a pluripolar set;*
(2) *$V_K^*$ is a locally bounded function. Furthermore in this case one has $V_K^* \in \mathcal{L}(\mathbb{C}^n)$.*

If $K$ is pluripolar then $V_K^* \equiv +\infty$, although it is not true in general that $V_K \equiv +\infty$.

It follows that the quantity $T_R(K) := exp(-sup\,\{\, V_K^*(z) \mid ||z|| \le R\,\})$ vanishes exactly on pluripolar sets (for any $R > 0$). This is how the *Siciak capacity* is introduced.

**Definition 2.1.40 (Siciak capacity).** *The quantity*

$$T(K) := T_1(K)$$

*is called the Siciak capacity of a relatively compact subset $K$ of $\mathbb{C}^n$.*

Of course the choice of any other positive number $R$ gives essentially equivalent capacity.

The last result we shall state is the mentioned connection between the two introduced capacities. These inequalities are due to Alexander and Taylor [AT]. We refer to [K4] for a short proof.

**Theorem 2.1.41 (Alexander-Taylor inequalities).** *If $K \subset \{\, z \mid ||z|| \le r\,\},\ r < R$ is a compact set, then there exist a constant $A(r) > 0$, dependent only on $r$, such that*

$$exp(-A(r)cap(K, B(0,R))^{-1}) \le T_R(K) \le exp(-2\pi cap(K, B(0,R))^{-1/n}),$$

*where $B(0,R)$ denotes the ball of radius $R$ centered at the origin.*

For more information regarding capacities, extremal fucntions and their corelations we refer to [K4] and [Kli].

2.1.5. *The Monge-Ampère operator and unbounded functions - Cegrell classes.* Recall that Bedford and Taylor results were proven only for locally bounded plurisubharmonic functions. So, a question arises whether for any $u \in PSH$ a positive Radon measure $MA(u)$ can be defined in such a way that the definition is coherent with the classical one for smooth functions and also with the Bedford and Taylor definition in the case of locally bounded functions. It is also desirable to preserve some of the basic featires of the operator known from the bounded case. Thus first we should fix the properties of the Monge-Ampère operator that we want to preserve. Note that any plurisubharmonic function can be locally approximated by decreasing sequence of smooth plurisubharmonic functions. In the bounded case the operator is continuous with respect to decreasing sequences, so it is natural to impose this kind of continuity in the general case either.

The example below (due to Kiselman) shows that such a definition is impossible for arbitrary plurisubharmonic functions:

**Example 2.1.42** (**Kiselman** [Ki1])**.** *Let* $u(z) = (-\log|z_1|)^{\frac{1}{2}}(|z'|^2 - 1)$ *where* $z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$*. Then* $u$ *is plurisubharmonic near* $0$*, but*

$$\int_{B(0,R)\setminus L} (dd^c u)^n = \infty,$$

*where* $L = \{z_1 = 0\}$ *and* $R > 0$ *is arbitrary (small) constant.*

This example leads to a natural problem of studying the class of plurisubharmonic functions for which a consistent definition is possible. Some partial results were known long ago, for example Sibony and Demailly have shown that the operator is well defined for functions bounded near the boundary of their domain of definition (see, for example, [Kli]). Systematical studies, however, started approximately 10 years ago.

The problem can be attacked in two ways: either to consider the problem locally, i.e. to define the Monge-Ampère operator locally (near a point) or to study functions in a fixed domain and control the behavior near the boundary of that domain. The first approach was used by Blocki [Bl4]. Below we state his main characterization result:

**Theorem 2.1.43.** *For any (nonnegative) plurisubharmonic functions the following conditions are equivalent:*

1. *$u$ has well defined Monge-Ampère measure, so that convergence of decreasing sequences implies weak-* convergence of the measures;*
2. *for any sequence of smooth plurisubharmonic functions $u_j$ decreasing towards $u$ the sequence of measures $(dd^c u_j)^n$ is weakly bounded;*
3. *for any sequence of smooth plurisubharmonic functions $u_j$ decreasing towards $u$ the sequences of measures*
$$|u_j|^{n-2-p} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \omega^{n-p-1}, \quad p = 0, 1, \cdots, n-2$$
*are weakly bounded;*
4. *there exists a sequence of smooth plurisubharmonic functions $u_j$ decreasing towards $u$, such that the sequences of measures*
$$|u_j|^{n-2-p} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \omega^{n-p-1}, \quad p = 0, 1, \cdots, n-2$$
*are weakly bounded;*

*($\omega := dd^c||z||^2 = 2\sum_{j=1}^n dz_j \wedge \overline{dz_j}$ is the canonical Kähler form in $\mathbb{C}^n$).*

We denote (temporarily) the class of plurisubharmonic functions satisfying the above conditions by $\mathcal{D}$, and whenever the domain $\Omega$ where we consider our functions is fixed the class will be denoted by $\mathcal{D}(\Omega)$.

The second approach (which was actually historically first) was developed by Cegrell (see [Ce2], [Ce3]). It turns out that the naural domains suitable for such a theory are the hyperconvex domains. Thus we assume throughout that the domain under consideration is hyperconvex. Below we list the basic notions in the field:

**Definition 2.1.44** (**The class** $\mathcal{E}_0$)**.** *The class of negative bounded plurisubharmonic functions $u$ on $\Omega$, such that $\lim_{z\to\zeta} u(z) = 0$ $\forall\zeta \in \partial\Omega$ and $\int_\Omega (dd^c u)^n < \infty$ is denoted by $\mathcal{E}_0(\Omega)$.*

These functions will be used as „basis" from which we shall generate unbounded functions with properly defined Monge-Ampère measure. Here and below if the domain considered is fixed, we shall simply write $\mathcal{E}_0 := \mathcal{E}_0(\Omega)$.

**Definition 2.1.45 (Cegrell classes).** *Let*

$$\forall p > 0 \;\; \mathcal{E}_p := \{u \in PSH(\Omega) |\; \exists u_j \in \mathcal{E}_0 : \; u_j \searrow u, \; j \to \infty, \; sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty\}.$$

*If additionally the sequence $u_j$ can be chosen so that it satisfies $sup_j \int_\Omega (dd^c u_j)^n < \infty$, then $u$ is said to belong to $\mathcal{F}_p$. If only $sup_j \int_\Omega (dd^c u_j)^n < \infty$ holds, then $u$ belongs to the class $\mathcal{F}$.*

*Finally if this condition holds only locally, i.e.*

$$\forall w \in \Omega \; \exists U_w \subset \Omega, \; U_w - \text{open} \; \exists g_w \in \mathcal{F}(\Omega) : g|_{U_w} = g_w|_{U_w},$$

*then $u$ belongs to the class $\mathcal{E}$.*

It turns out that functions in these classes have properly defined Monge-Ampère measures (see [Ce2]), which are simply the weak limit of the measures $(dd^c u_j)^n$ (this limit exists and is independent of the approximating sequence).

Note that all the defined classes are contained in $\mathcal{E}$. But to say that $\mathcal{E}$ is indeed the largest possible domain of definition of the Monge-Ampère operator, one has to compare it with the class $\mathcal{D}(\Omega)$. We recall the fundamental result proven by Blocki:

**Theorem 2.1.46** ([Bl4])**.** *Let $\Omega$ be hyperconvex domain. Then*

$$\mathcal{E}(\Omega) = \mathcal{D}(\Omega).$$

So, the "local" and "global" maximal domain of definition coincide.

The basic properties of functions from Cegrell classes can be found in [Ce3]. Below we present only these that will be used in our further study.

The first and probably the most important result is the comparison principle:

**Theorem 2.1.47 (Comparison principle in Cegrell classes).** *Let $u, \; v \in \mathcal{E}^p, \; p > 0$. Then*

$$\int_{\{u<v\}} (dd^c\, v)^n \leq \int_{\{u<v\}} (dd^c\, u)^n.$$

*The conclusion still holds in the classes $\mathcal{F}$ and $\mathcal{E}$ provided we further assume that the measure $(dd^c\, v)^n$ doesn't charge pluripolar sets i.e. $\forall A \subset \Omega$, $A$- borelean pluripolar set we have $(dd^c\, v)^n(A) = 0$ (in the case of the class $\mathcal{E}$ we have to make also the assumption that $\{u < v\} \Subset \Omega$) .*

The comparison principle is a basic tool in pluripotential theory also in the unbounded case. The result itself was proven (after several weaker versions) by Cegrell in [Ce3]. We would like to point out the somewhat non natural condition of lack of pluripolar charges. From measure theoretic point of view a complex hyperplane in $\mathbb{C}^n$ is undoubtedly much "larger" set than $\mathbb{R}^n \subset \mathbb{C}^n$ but our condition allows charging the second set and not the first one! It turns out, however, that this condition is not artificial as we shall see in our further studies, which shall justify its significance.

When working with the Monge-Ampère operator, one often has to deal with "mixed" measures i.e. terms of the type

$$dd^c\, u_1 \wedge dd^c\, u_1 \wedge \cdots \wedge dd^c\, u_n.$$

Here a very useful tool are the inequalities of Hölder type proven by Cegrell in [Ce3] for the class $\mathcal{F}$.

**Theorem 2.1.48.** *Let $u \in \mathcal{E}_0(\Omega),\ v_1, \cdots, v_n \in \mathcal{F}(\Omega)$. Then the following inequalities hold*

$$\int_\Omega -u(dd^c v_1) \wedge (dd^c v_2) \wedge \cdots \wedge (dd^c v_n) \le (\int_\Omega -u(dd^c v_1)^n)^{\frac{1}{n}} \cdots (\int_\Omega -u(dd^c v_n)^n)^{\frac{1}{n}};$$

$$\int_\Omega -u(dd^c v_1) \wedge (dd^c v_2) \wedge \cdots \wedge (dd^c v_n) \le$$

$$\le (\int_\Omega -v_1(dd^c v_1)^n)^{\frac{1}{n+1}} \cdots (\int_\Omega -v_n(dd^c v_n)^n)^{\frac{1}{n+1}} (\int_\Omega -u(dd^c u)^n)^{\frac{1}{n+1}}.$$

An important object that we shall often deal with is the integral of a function from the class $\mathcal{E}^1$ integrated against its own Monge-Ampère measure.

**Definition 2.1.49.** *Let $u \in \mathcal{E}^1(\Omega)$. The term*

$$\int_\Omega -u(dd^c u)^n$$

*will be called $\mathcal{E}^1$-energy of the function $u$.*

**Proposition 2.1.50.** *Let $u,\ v \in \mathcal{E}^1(\Omega),\ u \le v$. Then*

$$\int_\Omega -v(dd^c v)^n \le \int_\Omega -u(dd^c u)^n.$$

This shows that $\mathcal{E}^1$-energy of a function is controlled by $\mathcal{E}^1$-energy of any smaller function in $\mathcal{E}^1$.

2.1.6. *Additional results.* Often in applications instead of the inequalities from Theorem 2.1.48 one needs estimates in the opposite direction. This time, however, we shall estimate the measures themselves rather than their integrals. The results in this section are taken from [Di2].

Since the Monge-Ampère operator is simply

$$MA(u) := 4^n n! \det(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}) d\lambda,$$

one can use the theory of positive definite matrices in the study of this operator. We refer to [HJ] for the basic facts on positive definite Hermitian matrices. Now, following [Bl2] (see also [W1]), one can use concavity properties of such matrices to obtain pointwise estimates for the Monge-Ampère operator . In particular the following holds:

**Theorem 2.1.51.** *Let $u,\ v$ be bounded and smooth plurisubharmonic functions such that $(dd^c u)^n \ge f d\lambda$, $(dd^c v)^n \ge g d\lambda$, where $f$ and $g$ are smooth nonnegative functions and $d\lambda$ is the Lebesgue measure. Then*

$$(dd^c u)^k \wedge (dd^c v)^{n-k} \ge f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\lambda, \tag{2.1}$$

$$(dd^c(u+v))^n \ge (f^{\frac{1}{n}} + g^{\frac{1}{n}})^n d\lambda. \tag{2.2}$$

In the smooth case these inequalities were proved in [Ga]. In fact these inequalities are direct consequences of the arithmetic means - geometric means inequality, as can be seen if we simultaneously diagonalize $dd^c u$ and $dd^c v$ (as Hermitian forms, not as matrices).

Since we deal with general plurisubharmonic functions in Cegrell classes, (which are neither smooth nor even bounded) a question appears whether our inequalities (suitably understood) hold in this more general situation. In particular, we want to know whether the following holds:

*Let $u$, $v$ be psh functions in some Cegrell class (so, $(dd^c u)^n$ and $(dd^c v)^n$ are well defined). Let also $\mu$ be a positive measure and $f,\ g \in L^1(d\mu)$. Suppose that $(dd^c u)^n \geq f d\mu$, $(dd^c v)^n \geq g d\mu$ in the sense of measures. Is it true that (again in the sense of measures)*

$$(dd^c u)^k \wedge (dd^c v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu, \tag{2.3}$$

$$(dd^c(u+v))^n \geq (f^{\frac{1}{n}} + g^{\frac{1}{n}})^n d\mu. \tag{2.4}$$

We start our discussion with an example showing that in general these inequalities are false.

**Example 2.1.52.** *Let $u_k = \max\{\frac{1}{k}\log|z_1|, k^2\log|z_2|\}$, $v_k = \max\{\frac{1}{k}\log|z_2|, k^2\log|z_1|\}$, $k \in \mathbb{R}$, $k > 0$. It can be proved that both $u_k$ and $v_k$ belong to $\mathcal{E}$ for any $k > 0$ (one can consider, for example, the unit bidisc as the domain where both functions live).*

*Then*

$$(dd^c u_k)^2 = (2\pi)^2 \frac{k}{2}\delta_0,\ (dd^c v_k)^2 = (2\pi)^2\frac{k}{2}\delta_0$$

*but*

$$dd^c u_k \wedge dd^c v_k = (2\pi)^2 \frac{1}{2k^2}\delta_0,$$
$$(dd^c(u_k+v_k))^2 = (2\pi)^2(k+\frac{1}{k})\delta_0,$$

*where $\delta_0$ is the Dirac delta. In particular inequalities (2.3) and (2.4) both fail in this case.*

**Remark 2.1.53.** *This example is borrowed from Wiklund's paper* [W2], *where these functions were used in a different context. The computations below are based on the ideas shown in* [R1], [R2].

*Proof.* First let us compute $(dd^c u_k)^2$, ($(dd^c v_k)^2$ goes, by symmetry, the same way). Since $u_k \in \mathcal{E} = \mathcal{D}$, by Błocki's Theorem 2.1.43 it is enough to compute $(dd^c u_{k,j})^2$ with $u_{k,j} := \max\{u_k, -j\}$. Now proceeding as in [Bl3] we use the change of the variable

$$(x,y) \longrightarrow (\log|z_1|, \log|z_2|)$$

to confirm that

$$\int_{\mathbb{D}^2} (dd^c u_{k,j})^2 = (2\pi)^2 \int_{x\leq 0, y\leq 0} MA(\overline{u}_{k,j}),$$

where $MA$ is the real Monge-Ampère operator and $\overline{u}_{k,j}(x,y) := \max\{\frac{1}{k}x,\ k^2y,\ -j\}$. By Alexandrov's theorem (see [Al]) the latter integral is equal to the volume of the gradient image, i.e.

$$\int_{x\leq 0,y\leq 0} MA(\overline{u}_{k,j}) = \lambda(\nabla\overline{u}_{k,j}(\{x\leq 0, y\leq 0\})),\ \nabla\overline{u}_{k,j}(E) := \cup_{w\in E}\nabla\overline{u}_{k,j}(w),$$

where $\nabla\overline{u}_{k,j}(w) := \{t \in \mathbb{R}^n |\ u_{k,j}(w) + < s-w, t > \leq u_{k,j}(s),\ \forall s \in Dom\, u_{k,j}\}$.

At points where $u_{k,j}$ is smooth $\nabla\overline{u}_{k,j}(w)$ is a singleton set (the usual gradient of $u_{k,j}$), while at non-smooth points usually $\nabla\overline{u}_{k,j}$ is a larger set. Hence at points where $u_{k,j}$ is smooth and equal to $\frac{1}{k}x$ we get that $\nabla\overline{u}_{k,j}(w) = \{(\frac{1}{k},0)\}$. Analogously in the two other smooth regions $u_{k,j} = k^2y$ and $u_{k,j} = -j$ we get that $\nabla\overline{u}_{k,j}(w)$ is equal to $\{(0,k^2)\}$ and $\{(0,0)\}$, respectively. Note that the Lebesgue measure of the gradient image for this set is 0. Let now $w$ is a point where (for example) $\frac{1}{k}x = k^2 y > -j$. Then one easily computes the gradient image to be the line segment joining $(\frac{1}{k}, 0)$ and $(0, k^2)$. Analogously for the other points where two of the three functions considered in the maximum coincide the gradient image is a line segment joining the corresponding endpoints. Finally at the point$(-kj, \frac{-j}{k^3})$ (all three functions coincide), the gradient image will be the full triangle with vertices $(\frac{1}{k}, 0),\ (0, k^2),\ (0, 0)$.

The analysis above shows us that the total mass of $(dd^c\, u_{k,j})^2$ over the unit bidisc is equal to $(2\pi)^2 \frac{k}{2}$. Also if we fix a set $U \subset \mathbb{D}^2$ disjoint from the origin, its logarithmic image would not contain $(-kj, \frac{-j}{k^3})$ for $j$ large, hence the gradient image of that set will have zero Lebesgue measure. This shows that $(dd^c\, u_k)^2$ is concentrated at the origin and since we know the total mass we find that $(dd^c\, u_k)^2 = (2\pi)^2 \frac{k}{2} \delta_0$. Note that $(dd^c\, v_k)^2 = (2\pi)^2 \frac{k}{2} \delta_0$ by symmetry.

We are left to compute $dd^c\, u_k \wedge dd^c\, v_k$. But note that

$$2dd^c u_k \wedge dd^c\, v_k = (dd^c(u_k + v_k))^2 - (dd^c\, u_k)^2 - (dd^c\, v_k)^2,$$

so all we need to do is to compute $(dd^c(u_k + v_k))^2$. Note that

$$u_k + v_k = \max\{(k^2 + \frac{1}{k}) \log |z_1|,\ \frac{1}{k} \log |z_1 z_2|,\ (k^2 + \frac{1}{k}) \log |z_1|\}.$$

Arguing in the same way the gradient image of $\max\{\overline{u_k + v_k},\ -j\}$ is the (obtuse) rectangle with vertices (clockwise) $(0, k^2 + \frac{1}{k}),\ (\frac{1}{k}, \frac{1}{k}),\ (k^2 + \frac{1}{k}, 0),\ (0, 0)$ which has volume $k + \frac{1}{k^2}$. Hence as above $(dd^c(u_k + v_k))^2 = (2\pi)^2 (k + \frac{1}{k^2}) \delta_0$, and finally $dd^c\, u_k \wedge dd^c\, v_k = (2\pi)^2 \frac{1}{2k^2} \delta_0$. □

Note that in the above example the Monge-Ampère measures of the functions involved charge pluripolar sets. It turns out that this is precisely the obstruction for (2.3) and (2.4) to hold.

Below we state the main result in this section.

**Theorem 2.1.54.** *Let $\mu$ be a positive measure in a domain $\Omega$ that vanishes on all pluripolar sets. Let $u_1, u_2, \cdots, u_n \in PSH(\Omega)$ be plurisubharmonic functions with well defined Monge-Ampère operator. Let also $f_i,\ i = 1, \cdots, n$ be nonnegative functions integrable with respect to $\mu$. If*

$$(dd^c\, u_i)^n \geq f_i d\mu,\ \forall i = 1, \cdots, n$$

*then*

$$dd^c\, u_1 \wedge dd^c\, u_2 \wedge \cdots \wedge dd^c\, u_n \geq (f_1 f_2 \cdots f_n)^{\frac{1}{n}} d\mu$$

**Remark 2.1.55.** *We can alternatively prove such a result in the setting of germs of functions as in* [Bl4]*. To unify the two possible approaches we shall work locally (in a small ball). Also, for the sake of brevity, we shall work throughout the note with two functions $u$ and $v$ instead of the collection of $n$ functions. It will be explained how to get this general case.*

*Proof.* In [K3] the claimed statement was proved in the case when $\mu$ is the Lebesgue measure and both $u$ and $v$ are continuous. The idea was to approximate $u$ and $v$ by smooth plurisubharmonic functions and, using the inequality known from the smooth case, to obtain the same inequality in the limit of the approximation process. The delicate point is to show good enough convergence of the Monge-Ampère currrents of the approximants towards the measures associated to $u$ and $v$.

We shall follow a similar strategy. We shall find appropriate sequences $u_j,\ v_j$ for which Theorem 2.1.54 holds, and prove that they converge in a suitable way to $u$ and $v$

respectively ensuring the weak convergence of $(dd^c u_j)^k \wedge (dd^c v_j)^{n-k}$ towards $(dd^c u)^k \wedge (dd^c v)^{n-k}$. To illustrate the difficulties recall that (see example 2.1.26) the weak star convergence of $(dd^c u_j)^n$ towards $(dd^c u)^n$ (even if all the fuctions considered have the same boundary values) does not imply strong enough convergence (that is convergence in capacity) of $u_j$ towards $u$. On the other hand, taking for example convolutions would be enough for the convergence in capacity but the inequality for the approximants is unclear.

For this reason we introduce a special sequence of approximating measures of a fixed measure $\mu$:

Given a positive Radon measure on a bounded domain $\Omega \Subset \mathbb{C}^n$ we define its *canonical approximation* (see [K1]): Let *supp* $\mu$ be contained in a big cube $I$. Consider a subdivision $\mathcal{B}_k$ of $I$ into $3^{2kn}$ congruent semi open cubes $I_k^j$, $j = 1, \cdots, 3^{2kn}$. It is no loss of generality to assume $\mu(\cup_{I_k^j \in \mathcal{B}_k} \partial(I_k^j)) = 0$ (otherwise we can shift at each stage the boundaries a bit). Now define

$$\mu_k := \sum_j \frac{\mu(I_k^j \cap \Omega)}{dV(I_k^j \cap \Omega)} \chi_{I_k^j}\, dV, \tag{2.5}$$

where $\chi_{I_k^j}$ is the characteristic function of $I_k^j$. Of course $\mu_k$ is weak* convergent to $\mu$ and every term $\mu_k$ has a density in $L^\infty$ with respect to the Lebesgue measure.

Before we proceed we collect some results that we shall use later on:

**Theorem 2.1.56.** *Let $\Omega$ be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and let $f \in \mathcal{C}^\infty(\partial\Omega)$ be arbitrary. Let also $\mu$ be a positive measure on $\Omega$ with finite mass and compact support. Suppose $\mu$ satisfies the following condition for any $p > \frac{n}{n-1}$:*

*There is a constant $A = A(p)$ such that*

$$\int_\Omega (-\phi)^p d\mu \leq A\Big(\int_\Omega (-\phi)^p (dd^c \phi)^n\Big)^{\frac{p}{n+p}}$$

*for any $\phi \in \mathcal{E}_0$. Then:*

1. *There exist $u_k \in \mathcal{C}(\overline{\Omega})$ which solve the Dirichlet problem:*
$$\begin{cases} u_k \in PSH(\Omega) \cap C(\overline{\Omega}) \\ (dd^c u_k)^n = \mu_k \\ u_k = f \text{ on } \partial\Omega, \end{cases}$$
*where $\mu_k$ are the canonical approximants of $\mu$.*
2. *Define $u := (limsup_{k\to\infty} u_k)^*$. Then there is a subsequence of $\{u_k\}$ (which after renumbering we still denote by $\{u_k\}$) such that $u_k \to u$ in $L^1(d\lambda)$.*
3. *We have for this sequence that*
$$\sup_k \int_\Omega |-u_k| (dd^c u_k)^n < \infty$$
$$\lim_{k\to\infty} \int_\Omega |u - u_k| (dd^c u_k)^n = 0.$$

The proof of the first part may be found in [K1]. Other results follow from Theorems 5.1 and 7.7 and Lemmas 5.2, 5.3, 7.8 and 7.9 from [Ce2]. We would like to mention that the condition that all the functions have the same boundary values can be weakened. If, for example, the boudary values of $u_k$ form a sequence decreasing towards a bounded upper semicontinuous function (which will be the case we shall use later on) $u := (limsup_{k\to\infty} u_k)^*$ still makes perfect sense and, applying line by line the proofs from [Ce2] we get the finiteness and the convergence of the integrals in this situation too.

**Theorem 2.1.57.** *Suppose $u_j \in PSH(\Omega) \cap C(\overline{\Omega})$ is a sequence that converges to $u \in PSH(\Omega)$ in $L^1(d\lambda, \Omega)$. Suppose also all $u_k$ (and hence also $u$) have the same continuous boundary values, i.e. $\lim_{z\to\zeta} u_j(z) = f(\zeta)$ $\forall \zeta \in \partial\Omega$. If moreover $\lim_{k\to\infty} \int_\Omega |u - u_k|(dd^c u_k)^n = 0$ then $u_k$ converges to $u$ in capacity.*

This result is contained in the proof of Lemma 2.1 in [CK2]. Again we can carry the argument from [CK2] also if we let boundary values of $u_j$ to decrease (to be precise, since in the proof there boundary values are used only to ensure relative compactness of sets $\{u_j < u - a\},\ a > 0$, it is even better when boundary values of $u_j$ are bigger than those of $u$).

Now we go back to the proof of the theorem. Recall that we consider the case of two different functions in order to avoid technicalities in the notation.

First we consider the case of *bounded* $u$ and $v$.

Note that the claimed inequality is local, hence it suffices to prove it in a (small) ball $\mathbb{B}^n$, such that the functions $u,\ v$ are defined in a neighbourhood of it. Let $m_j,\ n_j$ be two sequences of smooth functions on $\partial\,\mathbb{B}^n$, decreasing to $u|_{\partial\,\mathbb{B}^n}$ and $v|_{\partial\,\mathbb{B}^n}$ respectively. Let $u_j,\ v_j$ solve

$$\begin{cases} u_j \in PSH(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n) \\ (dd^c\, u_j)^n = (dd^c\, u)^n_j \\ u_j|_{\partial\,\mathbb{B}^n} = m_j \end{cases}$$

$$\begin{cases} v_j \in PSH(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n) \\ (dd^c\, v_j)^n = (dd^c\, v)^n_j \\ v_j|_{\partial\,\mathbb{B}^n} = n_j. \end{cases}$$

We recall that $(dd^c\, u)^n_j$ is the canonical approximation of the measure $(dd^c\, u)^n$. By Theorem 2.1.56 such solutions exist.

Before we proceed we would like to point out some subtleties. If $u$ and $v$ were continuous, the sequences $m_j,\ n_j$ would be redundant (since we can work with merely continuous boundary data as well). This point causes some technical problems in the proof. Also we need here to use the canonical approximants for the measures on the right hand side instead of the measures themselves, for the following reason: The Dirichlet problem

$$\begin{cases} u_j \in PSH(\mathbb{B}^n) \cap L^\infty(\mathbb{B}^n) \\ (dd^c\, u_j)^n = (dd^c\, u)^n \\ u_j|_{\partial\,\mathbb{B}^n} = m_j \end{cases}$$

need not have a solution continuous up to the boundary.

**Proposition 2.1.58.** *Let $u_j,\ v_j$ be as above. Define $u := (\limsup_{j\to\infty} u_j)^*$, $v := (\limsup_{j\to\infty} v_j)^*$. Assume also that $u_j$ (resp. $v_j$) tend to $u$ (resp. $v$) in $L^1(d\lambda)$. Then we have*

$$(dd^c\, u_j)^k \wedge (dd^c\, v_j)^{n-k} \rightharpoonup (dd^c\, u)^k \wedge (dd^c\, v)^{n-k},\ \forall k \in \{1, \cdots, n\}.$$

*Proof.* Note that the inequality

$$\int_\Omega (-\phi)^p d\mu \le A(p)\Big(\int_\Omega (-\phi)^p (dd^c\, \phi)^n\Big)^{\frac{p}{n+p}},\ \phi \in \mathcal{E}_0(\Omega)$$

holds for any $p$ with a constant $A(p)$ dependent on $p$, if $\mu$ is the Monge-Ampère measure of a bounded plurisubharmonic function (this follows easily from the theory in [Ce2]). So, by Theorem 2.1.56 and the discussion after it, we have (after passing to an appropriate subsequences, which for the sake of brewity, will also be denoted by $u_j,\ v_j$), that

$$\lim_{k\to\infty}\int_\Omega |u-u_k|(dd^c u_k)^n=0,$$
$$\lim_{k\to\infty}\int_\Omega |v-v_k|(dd^c v_k)^n=0.$$

Now Theorem 2.1.57 (see also the remark after it) give us that $u_k$, and $v_k$ converge to $u$ and $v$ in capacity.

Now we are almost ready to approximate $(dd^c u)^k\wedge(dd^c v)^{n-k}$ by $(dd^c u_j)^k\wedge(dd^c v_j)^{n-k}$. Indeed Xing's theorem would give the claimed convergence provided $u_j$ and $v_j$ are locally uniformly bounded.

Unfortunately we do not have this information. However this difficulty can be bypassed by noticing that $u_j,\ v_j$ are uniformly bounded in $\mathcal{E}^1$ norm (see [Ce2] or [K4]): to show this take any $U\Subset\Omega$ with $cap(U,\mathbb{B}^n)<\epsilon$. Then by Theorem 2.1.48

$$\begin{aligned}
&\int_U (dd^c u_j)^k\wedge(dd^c v_j)^{n-k}\le\\
&\le\int_{\mathbb{B}^n}-h_{U,\Omega}(dd^c(u_j+U(0,-m_j)))^k\wedge(dd^c(v_j+U(0,-n_j)))^{n-k}\le\\
&\le(\int_{\mathbb{B}^n}-(u_j+U(0,-m_j))(dd^c(u_j+U(0,-m_j)))^n)^{\frac{k}{n+1}}\times\\
&\times(\int_{\mathbb{B}^n}-(v_j+U(0,-n_j))(dd^c(v_j+U(0,-n_j)))^n)^{\frac{n-k}{n+1}}(\int_{\mathbb{B}^n}-h_{U,\Omega}(dd^c h_{U,\Omega})^n)^{\frac{1}{n+1}}\le\\
&\le C^{\frac{n}{n+1}}cap(U,\Omega)^{\frac{1}{n+1}}\le C^{\frac{n}{n+1}}\epsilon^{\frac{1}{n+1}}
\end{aligned}$$

Where $C$ is the uniform $\mathcal{E}^1$ bound for $u_j,\ v_j$, $h_{U,\Omega}$ is the relative extremal function of $U$ and we have used the Hölder type inequalities, which is legal since $u_j+U(0,-m_j), v_j+U(0,-v_j)$ belong to $\mathcal{E}_0$ (see [Ce3]). The rigorous justification of the uniform $\mathcal{E}^1$ bound for the sequences is a bit technical and will be given in Lemma 2.1.59 below.

Now, again due to uniform $\mathcal{E}^1$ bounds and Theorem 2.1.48, we have

$$cap(\{u_j<-s\},\Omega)\le\int_\Omega-\frac{|u_j|}{s}(dd^c h_{\{u_j<-s\},\Omega})^n\le\frac{C}{s}$$

with $C$ independent of $j$ and $s$ (in fact much better estimates can be provided but these are satisfactory for our needs).

Fix $s$ big enough such that $cap(\{u_j<-s\},\Omega)\le\epsilon,\forall j$. Then for any test function $\chi$ we have

$$\begin{aligned}
&|\int_{\mathbb{B}^n}\chi((dd^c u_j)^k\wedge(dd^c v_j)^{n-k}-(dd^c u)^k\wedge(dd^c v)^{n-k})|\le\\
&\le|\int_{\{u_j\le -s\}\cup\{v_j\le -s\}}\chi((dd^c u_j)^k\wedge(dd^c v_j)^{n-k}-(dd^c u)^k\wedge(dd^c v)^{n-k})|+\\
&+|\int_{\mathbb{B}^n}\chi((dd^c max(u_j,-s))^k\wedge(dd^c max(v_j,-s))^{n-k}-(dd^c u)^k\wedge(dd^c v)^{n-k})|
\end{aligned}$$

But the first term is arbitrarily small by the argument above and the second term tends to 0 due to Xing's theorem. So, we obtained the desired result. □

**Lemma 2.1.59.** *There is an absolute constant $C$ independent of $j$ such that*

$$\int_{\mathbb{B}^n} -(v_j + U(0,-n_j))(dd^c(v_j + U(0,-n_j)))^n < C$$

*Proof.* Consider the function $g_j := U((dd^c v_j)^n, 0)$. From [Ce2] we know that $g_j \in \mathcal{E}_0$ and $(dd^c g_j)^n = (dd^c v_j)^n$. Hence by comparison principle applied to the pair $v_j, g_j + U(0, n_j)$ we get

$$v_j + U(0,-n_j) \geq g_j + U(0,n_j) + U(0,-n_j).$$

Let $h_j := U(0,n_j) + U(0,-n_j)$ By inequality from Proposition 2.1.50 we get

$$\int_{\mathbb{B}^n} -(v_j + U(0,-n_j))(dd^c(v_j + U(0,-n_j)))^n \leq \int_{\mathbb{B}^n} -(g_j + h_j)(dd^c(g_j + h_j))^n.$$

The last term can be decomposed into a sum of terms of the type

$$\binom{n}{m} \int_{\mathbb{B}^n} -(g_j + h_j)(dd^c g_j)^m \wedge (dd^c h_j)^{n-m},\ m \in \{0, \cdots, n\}.$$

Again by Cegrell inequalities such terms are controlled from above by some product of $\int_{\mathbb{B}^n} -g_j(dd^c g_j)^n$ and $\int_{\mathbb{B}^n} -h_j(dd^c h_j)^n$. But $h_j$ are uniformly bounded, while

$$\int_{\mathbb{B}^n} -g_j(dd^c g_j)^n = \int_{\mathbb{B}^n} -g_j(dd^c v_j)^n \leq \int_{\mathbb{B}^n} -(v_j + U(0,-n_j))(dd^c v_j)^n.$$

Now $U(0,-n_j)$ is uniformly bounded, $(dd^c v_j)^n$ have uniformly bounded total masses, and $\sup_j \int_{\mathbb{B}^n} -v_j(dd^c v_j)^n$ is finite by Theorem 2.1.56. Hence we have obtained the claimed uniform bound. □

Now we are ready to prove our main inequality in the case of bounded functions: Consider the canonical approximation as in Theorem 2.1.58. We have that

$$(dd^c u)^k \wedge (dd^c v)^{n-k} = \lim_{j\to\infty} (dd^c u_j)^k \wedge (dd^c v_j)^{n-k} \geq$$

$$\limsup_{j\to\infty} \sum_j \chi_{I_k^j} \frac{(\int_{I_k^j}(dd^c u)^n)^{\frac{k}{n}} (\int_{I_k^j}(dd^c v)^n)^{\frac{n-k}{n}}}{dV(I_k^j)} dV \geq$$

$$\geq \limsup_{j\to\infty} \sum_j \chi_{I_k^j} \frac{(\int_{I_k^j} f d\mu)^{\frac{k}{n}} (\int_{I_k^j} g d\mu)^{\frac{n-k}{n}}}{dV(I_k^j)} dV \geq$$

$$\geq \limsup_{j\to\infty} \sum_j \chi_{I_k^j} \frac{(\int_{I_k^j} f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu)}{dV(I_k^j)} dV = f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu,$$

where we have used the inequality known from the "Lebesgue measure case" and the Hölder inequality.

The case of $n$ different functions instead of just two goes in the same way. The only difference is that we must use the generalised Hölder inequality (for $n$ functions) instead of the classical one that we used above.

This result can be generalised to unbounded plurisubharmonic functions. We show below that our inequality remains true provided $\mu$ does not charge pluripolar sets. Since this is a purely local result we state it in terms of the Cegrell classes in a hyperconvex domain.

Let $u,\ v \in PSH(\Omega) \cap \mathcal{E}(\Omega)$ satisfy

$$(dd^c u)^n \geq f d\mu,\ (dd^c v)^n \geq g d\mu$$

assume moreover that $\mu$ does not charge pluripolar sets. Then we have the same inequality as in the bounded case.

Indeed, recall the following known inequality which is a special case of Demailly's inequality (see for example [KH]):

$$(dd^c \max(u, -j))^n \geq \chi_{\{u>-j\}}(dd^c u)^n \tag{2.6}$$

for every $u$ in $\mathcal{E}(\Omega)$. Now by monotone convergence and the result in the bounded case we obtain

$$(dd^c u)^k \wedge (dd^c v)^{n-k} = lim_{j\to\infty}(dd^c \max(u, -j))^k \wedge (dd^c \max(v, -j))^{n-k} \geq$$
$$\geq limsup_{j\to\infty}(\chi_{\{u>-j\}}f)^{\frac{k}{n}}(\chi_{\{v>-j\}}g)^{\frac{n-k}{n}}\mu$$

the last term converges to $f^{\frac{k}{n}}g^{\frac{n-k}{n}}\mu$ (because $\mu$ does not charge the pluripolar set $\{u = -\infty\} \cup \{v = -\infty\}$), which proves the claim. □

2.2. **Complex geometry.** Recall that a complex manifold is an (even dimensional) manifold endowed with the complex structure, i.e. the transition maps between charts are holomorphic. We shall restrict our attention to the compact complex manifolds, although some of the concepts considered will be independent of the compactness assumption. So, unless converse is explicitly stated, throughout this section a manifold will mean compact manifold.

2.2.1. *Kähler manifolds: definitions and examples.* Let us fix a complex manifold $X$ and let $n = dimX$. In a local chart there are plenty of smooth positive $(1,1)$-forms. By a partition of unity one can patch together a collection of such local $(1,1)$-forms and thus produce a global smooth positive $(1,1)$-form. Such a form induces a Hermitian metric on the tangent bundle, hence is called *Hermitian* form.

Locally one may also choose a *closed* Hermitian form (for example if $z = (z_1, \cdots, z_n)$ are local coordinates then $dd^c\,||z||^2$ is such a local form). The question whether a *global* closed Hermitian form can be found leads to the concept of *Kähler manifolds*:

**Definition 2.2.1 (Kähler manifold).** *A complex manifold $X$ is called Kähler if there exists a global positive and closed $(1,1)$-form. Such a form is called Kähler form.*

Below we list some basic examples of Kähler manifolds:

**Example 2.2.2 (The projective space $\mathbb{P}^n$).** *Consider the set $\mathbb{P}^n$ of all complex lines passing through $0$ in $\mathbb{C}^{n+1}$. This set can be endowed with the (complex) manifold structure by using the natural projection from $\mathbb{C}^{n+1}\setminus\{0\}$ onto it. In order to define a Kähler form on $\mathbb{P}^n$ we consider the form*

$$dd^c\,log(|Z_0|^2 + \cdots + |Z_n|^2),$$

*where $Z_i$ are the coordinates in $\mathbb{C}^{n+1}\setminus\{0\}$. Note that, when restricting to a complex line through $0$, the form is invariant (because the function $\lambda \to log|\lambda|$ is harmonic on $\mathbb{C}\setminus\{0\}$). Thus it descends onto a closed positive $(1,1)$-form on $\mathbb{P}^n$. The constructed form is called Fubini-Study (Kähler) form and is often denoted by $\omega_{FS}$.*

**Remark 2.2.3.** *The coordinates $Z_0, \cdots, Z_n$ are called homogeneous coordinates and the $(n+1)$-tuple $[Z_0 : \cdots : Z_n]$, understood as a class modulo multiplications the coordinates by a complex number (the same for each coordinate) are convenient "coordinates" for working on projective spaces.*

Note that the restriction of a closed Hermitian $(1,1)$-form onto a complex submanifold is also closed Hermitian. Hence, we get the following proposition:

**Proposition 2.2.4.** *Any complex submanifold of a Kähler manifold is Kähler. In particular, all submanifolds of* $\mathbb{P}^n$ *are Kähler. Such manifolds are called projective or algebraic.*

Another standard construction of producing new manifolds form old ones is the Cartesian product. It is standard do verify that for Kähler manifolds $(X, \omega_X)$, $(Y, \omega_Y)$ the Cartesian product $X \times Y$ can be endowed with the Kähler form $\omega := \pi_X^* \omega_X + \pi_Y^* \omega_Y$, where $\pi_Z$ denotes the projection onto $Z$. Thus we get the following proposition:

**Proposition 2.2.5.** *A Cartesian product of two Kähler manifolds is Kähler.*

Another class of examples consists of *complex tori*:

**Example 2.2.6 (Complex torus).** *Let* $\Lambda$ *be a lattice in* $\mathbb{C}^n$*. Then the Kähler form with constant coefficients* $\omega := \sum_{j=1}^n i dz_j \wedge d\bar{z}_j$ *descends onto the quotient* $\mathbb{C}^n / \Lambda$ *(which topologically is a* $2n$*-dimensional torus). So, the quotient carries a Kähler form, hence is Kähler.*

**Remark 2.2.7.** *It can be proved (see* [De3] *for details) that not every complex torus is projective (projectivity depends on the way we choose the lattice* $\Lambda$*, although all these manifolds are homeomorphic to each other). Thus the Kähler manifolds form a strictly larger class than the projective manifolds.*

In order to make the discussion complete we give an example of non-Kähler complex manifold.

**Example 2.2.8 (Hopf surface).** *Let* $\phi : \mathbb{C}^2 \setminus \{0\} \ni z \to 2z \in C^2 \setminus \{0\}$*. One can verify that the group* $< \phi >$ *generated by the automorphism* $\phi$ *(of* $\mathbb{C}^2 \setminus \{0\}$*) acts properly discontinuously, hence the quotient* $\mathbb{C}^2 \setminus \{0\} / < \phi >$ *has the structure of a complex manifold. This manifold is called Hopf surface. It can be proved (see, for example,* [De3]*) that (for topological reasons) the Hopf surface does not admit any Kähler structure.*

For more information regarding Kähler geometry we refer to [De3], [GH] and [T].

2.2.2. *Algebraic geometry - divisors and bundles.* For this section we assume basic knowledge of the theory of several complex variables (holomorphic and meromorphic functions, analytic sets, etc.) We refer to [GH] for a much more detailed discussion of these topics.

We begin with the notion of a divisor:

**Definition 2.2.9 (Divisor).** *If* $X$ *is a complex manifold (non necessarily compact or Kähler), a divisor on* $X$ *is a locally finite formal linear combination*

$$D = \sum_i a_i V_i,$$

*where* $a_i$ *are constants, while* $V_i$ *are ireducible analytic hypersurfaces in* $X$ *(here locally finite means that any point has a neigbourhood such that there are only finite number of* $V_i$ *passing through that neighbourhood. The space of all divisors on* $X$ *form a natural Abelian group denoted by* $Div(X)$*.*

**Remark 2.2.10.** *Of course on compact manifolds "locally finite" is the same as finite. Note also that hypersurfaces* $V_i$ *are allowed to be singular. Irreducibility is a technical assumption made in order to have uniqueness in the formal sum representation. Indeed, if* $V$ *is a reducible hypersurface, then* $V = \cup_{i=1} W_i$*, where* $W_i$ *are irreducible components of* $V$ *(it is a standard fact that the decomposition is locally finite). And thus one can use instead of* $V$ *the formal sum* $\sum_i W_i$*.*

Especially interesting from algebro-geometric point of view are the *effective divisors.*

**Definition 2.2.11 (Effective divisor).** *A divisor $D$ is called effective if*

$$D = \sum_i a_i V_i,$$

*with all $a_i \geq 0$. (by uniqueness of the decomposition the notion is well defined). We denote effectiveness of a divisor by writing simply $D \geq 0$.*

Since hypersurfaces are (locally) the zero sets of holomorphic functions, below we discuss the links between these objects.

**Definition 2.2.12 (Order of a function).** *Let $g$ be a local holomorphic function given in a neighbourhood of a point $p \in V \subset X$. The order $ord_{V,p}(g)$ of $g$ along $V$ at $p$ is the largest integer $a$ such that (locally near $p$) the defining function $f$ of $V$ raised on power $a$ divides $g$.*

**Remark 2.2.13.** *It is known that in the case of germs of holomorphic functions if $u$ and $v$ are relatively prime at $p$ the same holds in a neigbourhood of $p$ (see, for example* [GH]*). This shows that on the regular part of $V$ the order is locally constant, and if $V$ is irreducible, it is a constant function in the domain of definition of $g$ intersected with $V$. So, we denote it in this case simply by $ord_V(g)$.*

**Observation 2.2.14.** *With the obvious assumptions on the domains of definition of $f$ and $g$ we have*

$$ord_V(fg) = ord_V(f) + ord_V(g).$$

If $f$ is a global meromorphic function (such functions may exist even if $X$ is compact) then locally one can write $f = \frac{g}{h}$ with holomorphic relatively prime $g$ and $h$. If $V$ is irreducible we define

$$ord_V(f) := ord_V(g) - ord_V(h).$$

**Definition 2.2.15.** *For a meromorphic function $f$ we define the divisor $(f)$ by*

$$(f) = \sum_V ord_V(f)V,$$

*(the sum is taken over all irreducible hypersurfaces for which $ord_V(f) \neq 0$). Analogously we define the divisor of zeroes by*

$$(f)_0 = \sum_V ord_V(g)V,$$

*(the definition is independent of the choice of local functions $g$), and the divisor of poles by*

$$(f)_\infty = \sum_V ord_V(h)V,$$

*(again, this is well defined). Clearly*

$$(f) = (f)_0 - (f)_\infty.$$

We postpone for a while the discussion of divisors in order to introduce the concept of a (holomorphic) vector bundle. As we shall see below, there are natural correspondences between divisors and holomorphic line bundles (i.e. vector bundles of rank 1).

Intuitively a holomorphic vector bundle (of rank $r$) locally looks like a Cartesian product of a piece of the manifold and $\mathbb{C}^r$, and these local Cartesian products are glued together to produce a smooth structure which in one direction looks like $\mathbb{C}^r$ (the fiber or the vertical direction), while in the other (horizontal) direction looks like the manifold under consideration.

Herebelow we give the formal definition:

**Definition 2.2.16 (Holomorphic vector bundle).** *Let $X$ be a complex manifold of dimension $n$ (non necessarily compact or Kähler, although we shall only be interested in the compact Kähler case). A holomorphic vector bundle $V$ of rank $r$ is a complex manifold of dimension $n+r$, such that the following holds:*

1. *There is a mapping $\pi : V \to X$, called the bundle projection such that for every $x \in X$ the set $\pi^{-1}(x)$ (the fiber over $x$) is a complex linear space of dimension $r$,*
2. *$X$ can be covered by open sets $U_\alpha$ in such a way that $\pi|_{\pi^{-1}(U_\alpha)}$ is biholomorphic to $U_\alpha \times \mathbb{C}^r$ (such a covering is called local trivialization),*
3. *if $\theta_\alpha : \pi|_{\pi^{-1}(U_\alpha)} \to U_\alpha \times \mathbb{C}^r$ is the local trivializing biholomorphism then for any $x \in U_\alpha \subset X$ the mapping $\theta_\alpha|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{C}^r$ is a $\mathbb{C}$-linear isomorphism, (recall that the linear structure on the fiber $\pi^{-1}(x)$ is induced from the first point),*
4. *for every $\alpha$, $\beta$ the map*
$$\theta_{\alpha\beta} = \theta_\alpha \circ \theta_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$
*is of form*
$$(z, \zeta) \to (z, g_{\alpha,\beta}(z)(\zeta)),$$
*for some complex matrix $g_{\alpha,\beta}(z)$ that varies holomorphically in $z$, (here by $g_{\alpha,\beta}(z)(\zeta)$ we mean simply the action of the matrix $g_{\alpha,\beta}(z)$ on the vector $\zeta$),*
5. *the matrices satisfy the following cocycle relation*
$$g_{\alpha,\beta}(z) g_{\beta,\gamma}(z) g_{\gamma,\alpha}(z) = I_r,$$
*for any $z \in U_\alpha \cap U_\beta \cap U_\gamma$ ($I_r$ is the identity matrix of rank $r$).*

A basic example is the tangent bundle for any complex manifold. One can verify that the definition above is in fact modelled on this example: the fiber corresponds to a tangent space at a point, the trivialization corresponds to an atlas of charts and the diffeomorphisms $\theta_\alpha$ are simply the differentials on the tangent spaces.

Another example is the trivial bundle $X \times \mathbb{C}^r$.

In fact the bundle is uniquely defined (mofulo isomorphism) by the collection of the trivializing sets and the transition matrices $g_{\alpha,\beta}$ satisfying the cocycle property.

The notion that corresponds to the (global) vector fields in the tangent bundle is the noton od a *section*:

**Definition 2.2.17 (Section).** *A (holomorphic) section of a holomorphic vector bundle $V$ over a manifold $X$ is a mapping $s : X \to V$, such that $\pi \circ s = id_X$.*

Intuitively, a section is a choice of a vector in each fiber made in a holomorphic way. Of course, in the case of a trivial bundle this is nothing but a (usual) holomorphic mapping. Since there are no (nonconstant) holomorphic functions over a compact manifold we see that nontrivial global sections may not exist: their existence depends on the particular geometric situation.

Especialy interesting is the case of a line bundle:

**Definition 2.2.18 (Line bundle).** *A (holomorphic) line bundle is a (holomorphic) vector bundle of rank* 1.

In this case instead of holomorphic matrices one has to consider just local holomorphic functions. By the cocycle condition these functions are nowhere zero.

As in the general case *any* collection of local holomorphic functions satisfying the cocycle property gives rise to a line bundle.

Using this fact one can endow the set of all line bundles (modulo isomorphisms) with the natural structure of a group by introducing the operations in the following way:

If $L$ and $L'$ are line bundles given in a local trivialization (one can always shrink the trivializing sets, so that we can assume these sets are identical for both bundles) by the holomorphic functions $g_{\alpha,\beta}$ and $g'_{\alpha,\beta}$ respectively then we define

$$L \otimes L' \text{ and } L^*$$

as the line bundles given in local trivialization by functions $g_{\alpha,\beta}g'_{\alpha,\beta}$ and $g^{-1}_{\alpha,\beta}$ respectively.

The set of all line bundles endowed with these operations is an Abelian group called the Picard group of $X$ and we denote it by $Pic(X)$.

**Remark 2.2.19.** *the cocycle relation means that* $\{ g_{\alpha,\beta} \}$ *form a chain in the Cech cohomology of the sheaf* $\mathcal{O}^*$*. Therefore, from cohomological point of view,* $Pic(X)$ *is isomorphic to the cohomology group* $\mathcal{H}^1(X, \mathcal{O}^*)$*. We shall not use this relation. Instead, we refer to* [De3] *or* [GH] *for a discussion on these topics.*

Now we turn back to divisors: Recall that any divisor $D$ with integral coefficients has a local meromorphic defining function $f_\alpha$ in each sufficiently small open coordinate ball $U_\alpha$ such that $f_\alpha(z) \neq 0$ if $z \in U_\alpha \setminus \{ D \}$ ($\{ D \}$ is the geometrical support of $D$) and $f_\alpha$ vanishes (or has a pole) of order exactly equal to the coefficients in the expansion of the divisor. But then the functions $\frac{f_\alpha}{f_\beta}$ defined on $U_\alpha \cap U_\beta$ are holomorphic, nowhere vanishing and satisfy the cocycle property. Hence they define a bundle (called the associated bundle) which we denote by $[D]$.

It follows that

$$[D + D'] = [D] \otimes [D']$$
$$[-D] = [D]^*$$

Therefore the mapping

$$Div(X) \ni D \to [D] \in Pic(X)$$

is a group homomorphism.

It can be proved (see [GH]) that in the special case of *projective* manifolds this homomorphism is actually an isomorphism. Hence in this case there is virtually no difference between the analysis of divisors and line bundles. In the general case, however, it is not true that this homomorphism is an epimorphism - for example there are complex tori for which this is violated.

Below we discuss the central topic in the theory of divisors - the intersection numbers.

Assume, for simplicity, that $X$ is a complex surface (i.e. $dimX = 2$). Let $D$ and $E$ be two hypersurfaces which itersect transversally at each $x \in D \cap E$ and moreover both $D$ and $E$ are smooth near any such $x$. In such a case we define the intersection number of the associated divisors (denoted also by $D$ and $E$) as

$$D.E := \sharp \{ x | \ x \in D \cap E \} .$$

The notion easily generalizes to general divisors by linearity, provided one can define it on *any* two hypersyrfaces.

To achieve this we need the following two important facts which we state without proofs (which can be found, for example, in [GH]). Let us first recal the notion of homology:

**Definition 2.2.20.** *Two hypersurfaces $C_1$ and $C_2$ (non necessarily complex) are homologous if $C_1 - C_2$ is a boundary of a cycle of real dimension higher by one unit. More explicitly $C_1$ and $C_2$ are homologous if there exists a finite set of (piecewise smooth) real manifolds $D_i$ of real dimension $dim_{C_1} + 1 = dim_{C_2} + 1$, such that $\sum_i \partial D_i = C_1 - C_2$.*

**Theorem 2.2.21.** *For any two hypersurfaces $D$ and $E$ one can find hypersurfaces $D'$ and $E'$ homologous to $D$ and $E$ respectively, such that $D'$ and $E'$ intersect transversally and are smooth near any intersection point.*

**Theorem 2.2.22.** *If $D$ is homologous with $D'$ and they both satisfy the transversality assumptions with respect to $E$ then*

$$D.E = D'.E.$$

These two results show that for any two hypersurfaces it is enough to find transversal elements in their homology classes and define the intersection with their aid.

**Observation 2.2.23.** *If two divisors $D$, $E$ have disjoint supports then $D.E = 0$.*

**Remark 2.2.24.** *The defintion would suggest that the intersection number of two effective divisors is always nonnegative. This is indeed true if in the decomposition the divisors do not have common irreducible components. In fact on some surfaces there exist curves $C$ such that $C.C < 0$. Such curves are often called exceptional divisors and they are an important object of studies in algebraic geometry - see* [GH] *and* [La] *for details.*

On a general manifold of dimension $n$ the intersection can be defined in the same way as a $n$-linear mapping

$$Div(X)^n \ni (D_1, \cdots, D_n) \to D_1. \cdots .D_n.$$

Even more generally, one can define intersections for any varieties provided the dimensions are choosen in a way that generically the geometrical intersections should be discrete. A particular important example is an intersection of a divisor with a curve.

For more information on general properties of divisors we refer to [GH] and [La].

2.2.3. *Special divisors - ample, nef and big divisors.* In the previous subsections we have defined the notion of an effective divisor. In the quest for more subtle positivity notions many special classes of divisors were considered. In fact questions regarding (kinds) of positivity form one of the central topics in modern algebraic geometry. For us, however, these notions will only serve as an explanation of the geometrical background in the later charpters. Hence, in order to stick to the main line, we shall only sketch the ideas. For an extensive discussion of these concepts (furnished by many examples) we refer to [La] and [GH]. In this subsection we assume that our manifolds are *projective* - for otherwise many of the discussed notions (for example the notion of a nef divisor) would have to be defined in a different (more sophisticated) way.

We begin our discusion with the notion of *complete linear system.*

**Definition 2.2.25 (Complete linear system).** *Let $D$ be a divisor on a manifold $X$. A linear system of $D$ (denoted by $|D|$) is the set of all effective divisors $D'$ of the form $D' = D + (f)$ for some meromorphic function $f$.*

Below, for the sake of brewity, we shall call such an object a linear system.

**Remark 2.2.26.** *We should emphasize that $[D'] = [D]$ for any such divisors $D'$ and $D$. If $D = \sum a_i V_i$, then alternatively the linear system is detetrmined by all meromorphic functions $g$ with $ord_{V_i}(g) \geq -a_i$ at each hypersurface $V_i$.*

We should mention that the size of a linear system (i.e. the amount of its elements) depends on the manifold $X$ and on the particular choice of $D$ - intuitively the more positive $D$ is the larger the linear system $|D|$ should be.

This notion is heavily linked with the global (holomorphic) sections of the associated bundle $[D]$. Indeed, it is easily seen that an effective divisor $D'$ can be written as $(g)$ for a *holomorphic* section $g$ of the bundle $[D'] = [D]$ (here, by an abusement of notation, by $(g)$ we mean the divisor cut out by the zeroes of the local holomorphic functions which define the section $g$).

Note that a general line bundle may not have any global (nonzero) holomorphic section at all. From now on, in order to avoid trivialities, we assume thet such a section exists for the considered bundle $[D]$. The space of global sections clearly form a (complex) vector space and, by a classical result in algebraic geometry, this space is always finite dimensional. Thus one can fix a base for the space of global sections consisting of $s_1, \cdots, s_k$, where $k$ is the dimension of this space. This is the way we get the notion of the *associated map*:

**Definition 2.2.27 (Associated map).** *Let $D$ be a divisor in $X$. The meromorphic mapping*

$$\phi_{|D|} : X \ni z \dashrightarrow [s_1(z) : \cdots : s_k(z)] \in \mathbb{P}^{k-1}$$

*is called the associated map of the divisor $D$ (or the line bundle $[D]$). Here $[Z_1 : \cdots : Z_k]$ denotes the homogeneous coordinates in $\mathbb{P}^{k-1}$. The symbol $\dashrightarrow$ is used in order to emphasize that the mapping is meromorphic (in particular it need not be everywhere defined).*

Observe that the map depends on the particular choice of the basis. Hovewer any two bases differ merely by an isomorphic transformation, hence images of a manifold by two such asociated maps generated by two bases differ only by an isomorphism.

Since this is a canonical map into some projective space it is interseting to know when the following two properties are satisfied:

- under what assumptions $\phi_{|D|}$ is actually a holomorphic mapping,
- if so, when $\phi_{|D|}$ is an embedding?

This was the motivation for the notion of a *(very) ample divisor*:

**Definition 2.2.28 ((Very) ample divisor).** *A divisor $D$ is called very ample if the associated mapping $\phi_{|D|}$ is a (holomorphic) embedding. A divisor $D$ is called ample if $mD$ is very ample for some $m \in \mathbb{N}$.*

**Remark 2.2.29.** *In order to understand the philosophy of these notions, note that whenever $s$ is a section of $[D]$, then $s^m$ is a section of $[D]^m$. Thus the number of global sections of an ample line bundle is not decreasing when we take a multiple of it, and one can hope that some new sections may emerge, which, intuitively, contain more geometrical information and thus lead to better properties of the map $\phi$.*

Up to now the definitions were purely geometric. There are however important links between ampleness and positivity. In fact the role of ample divisors is central in the classical algebraic geometry. We shall not go into details (instead we refer to [De3], [La] or [GH]). We just give an example of such a link, which would perhaps enlight a bit the further developments in this subsection:

**Theorem 2.2.30 (Grauert criterion).** *Let $X$ be a complex surface (i.e. $dim X = 2$). A divisor $D$ on $X$ is ample if and only if $D^2 > 0$ and for every effective divisor (alternatively* *for every irreducible complex curve) $C$ one has $D.C > 0$. In higher dimensions there is an analogous result (called Nakai criterion).*

Next we shall discuss the notion of a *nef* divisor. These are central in classification theory in algebraic geometry (see [La]).

**Definition 2.2.31 (Nef divisor).** *Let $X$ be a projective manifold. A divisor $D$ on $X$ is called nef if for every irreducible curve we have $D.C \geq 0$.*

**Observation 2.2.32.** *The nef divisors form a convex cone in $Div(X)$.*

**Remark 2.2.33.** *Contrary to the intuition an effective divisor may not be nef - for reasons similar to the ones explained in Remark 2.2.24.*

**Remark 2.2.34.** *In dimension two the definition is equivalent to the following one: $D$ is nef if for any effective divisor $C$ we have $D.C \geq 0$. Thus, in a way, the nef cone is dual to the cone of effective divisors.*

**Observation 2.2.35.** *Any ample divisor is nef. Also a limit (in $Div(X)$) of ample divisors is nef. In fact, the converse also holds - the closure of the ample cone is equal to the nef cone.*

Another fundamental notion is the one of a *big* divisor. In order to avoid defining too many suplementary geometrical notions we give a somewhat desriptive definition:

**Definition 2.2.36 (Big divisor).** *Let $D$ be a divisor in $X$. For each $m \in \mathbb{N}$ consider the linear series generating $|mD|$ and the associated meromorphic map $\phi_{|mD|}X \dashrightarrow \mathbb{P}^{N(m)}$. $D$ is called big if the image of $X$ for some $m$ has maximal dimension (equal to $dimX$).*

**Observation 2.2.37.** *Note that this notion is in a way modelled on the embedding property of (very) ample divisors. Unlike this case, however, the mapping is assumed to be merely meromorphic. In particular there may exist points of indeterminacy (called base locus), where the mapping is not well-defined.*

Let us state two important results concerning big divisors (we refer to [La] for a proof):

**Theorem 2.2.38.** *If $D$ is a nef divisor then it is big in and only if $D^n > 0$.*

The second one is an important theorem of Iitaka which gives us more information about the mapping from the definition of big divisors:

**Theorem 2.2.39 (Iitaka fibration theorem).** *For a big divisor $D$ the associated map $\phi_{|mD|}X \dashrightarrow \mathbb{P}^{N(m)}$ is birational (that is, bimeromorphic) onto its image for some $m \in \mathbb{N}$.*

2.2.4. *Canonical bundle.* As we have observed the notion of a vector bundle was modelled on a particular case i.e. on the tangent bundle. In fact the tangent bundle has many properties that makes it worth studying. Perhaps the most important is its naturality i.e. it is naturally associated to *any* complex manifold.

Since our main interest is in the line bundles, a question appears whether there is a similar universal construction of a line bundle working for arbitrary manifolds. Thus we are led to the notion of a *canonical bundle*:

**Definition 2.2.40 (Canonical bundle).** *Let $X$ be a complex manifold of dimension $n$. The line bundle*

$$K_X := \Lambda^n T^*_X$$

*is called the canonical line bundle. Here $T^*_X$ denotes the dual of the tangent bundle, and $\Lambda^n$ is the (fiberwise) $n$-th exterior power.*

Properties of the canonical bundle are important in the classification of manifolds. Later on we will be interested in the following special cases:

- $K_X$ is ample (strictly speaking, the divisor associated to it by the isomorphism between $Div(X)$ and $Pic(X)$ is ample).
- $K_X$ it the trivial line bundle.
- $-K_X$ is ample.
- $K_X$ is big and/or nef.

2.2.5. *Chern classes.* Let us fix a divisor $D = \sum a_i V_i$. One can naturally associate with $D$ a current of integration according to the formula

$$\eta_D(\phi) := \sum a_i \int_{V_i} \phi,$$

where $\phi$ is a test form of bidegree $(n-1, n-1)$. Thus $\eta_D$ is a $(1,1)$-real current (which is positive if $D$ is effective). By $\{\eta_D\}$ we denote the De Rham cohomology class of this current. It follows from general theory (see [De3]) that this cohomology class can be represented (on a Kähler manifold) by a smooth $(1,1)$-form.

**Definition 2.2.41 (First Chern class).** *Let $X$ be a projective manifold. The first Chern class is the De Rham cohomology class in $H^2_{DR}(X, \mathbb{R})$ which is represented by the current associated to the divisor corresponding to the anticanonical line bundle $-K_X$. We denote the first Chern class of $X$ by $c_1(X)$.*

We say that the first Chern class is positive (which we denote by $c_1(X) > 0$) if we can find a Kähler form representing it. If one can find a Kähler representative for $-c_1(X)$ then the Chern class is negative (this is denoted by $c_1(X) < 0$). If $c_1(X)$ is the zero cohomology class then we simply write $c_1(X) = 0$.

In all the cases listed above the Chern class is said to be *definite*. In fact this is quite a special situation (recall that the coefficients of a divisor might not be of constant sign) and in dimension 2 these manifolds are classified (see [T] for more details). Intuitively a generic manifold will have an indefinite first Chern class.

Actually, there is another more analytic way to construct the first Chern class (see [T]) as a trace of the curvature tensor, however the algebraic definiton is satisfactory for our needs.

## 3. The Monge-Ampère operator on Kähler manifolds

### 3.1. Definitions.

3.1.1. *$\omega$-psh functions - basic definitions.* Given a compact complex manifold, there are no nonconstant plurisubharmonic function on it, for any such function will violate the maximum principle. However the space of positive $(1,1)$-currents may be large. Therefore we introduce a class of functions which are locally plurisubharmonic modulo some smooth function. A number of different names for this class of functions have appeared in the literature. They became known as quasiplurisubharmonic or admissible functions. We shall stick to the name $\omega$-plurisubharmonic ($\omega$-psh for short).

Let $X$ be compact $n$-dimensional Kähler manifold equipped with fundamental Kähler form $\omega$ given in local coordinates by

$$\omega = \frac{i}{2} \sum_{k,j=1}^{n} g_{k\bar{j}} dz^k \wedge d\bar{z}^j.$$

We assume that the metric is normalized so that

$$\int_X \omega^n = 1.$$

**Definition 3.1.1** (**$\omega$-plurisubharmonic functions**). *The class of $\omega$-plurisubharmonic functions ($\omega$-psh for short) is defined by*

$$PSH(X,\omega) := \{\phi \in L^1(X,\omega) : dd^c\phi \geq -\omega,\ \phi \in \mathcal{C}^{\uparrow}(X)\},$$

*where, as usual $d = \partial + \overline{\partial}$, $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$ and $\mathcal{C}^{\uparrow}(X)$ denotes the space of upper semi-continuous functions.*

Later on, for the sake of brevity, we shall often use the handy notation $\omega_\phi := \omega + dd^c\,\phi$.

Since the Bedfor-Taylor definition of the complex Monge-Ampère operator is local, one can also define this operator in the Kähler manifold setting for bounded $\omega$-psh functions:

**Definition 3.1.2** (**The Monge-Ampère operator**). *Let $\phi$ be a bounded $\omega$-psh function on an $n$-dimensional Kähler manifold $(X,\omega)$. Then locally $\omega_\phi = dd^c\,u$, for some local plurisubharmonic function $u$ and thus the Monge-Ampère operator*

$$\omega_\phi^n := \underbrace{\omega_\phi \wedge \cdots \wedge \omega_\phi}_{n-times}$$

*is locally (hence globally) well defined as a Borel measure.*

In the paper [K3] Kołodziej have defined a new capacity on a compact Kähler manifold which is modelled on the Bedford-Taylor relative capacity from the 'flat' setting.

**Definition 3.1.3** (**Relative capacity**). *The quantity*

$$cap_\omega(K) := \sup\{\int_K (\omega_\psi)^n | \psi \in PSH(X,\omega),\ 0 \leq \psi \leq 1\}$$

*is called the relative capacity of the (Borelean) set $K$.*

Analogously to the flat case, one can consider the convergence with respect to relative capacity, first used in the context of Kähler manifolds in [K3].

**Definition 3.1.4** ([K3]). *We say that a sequence $\phi_j \in PSH(X,\omega)$ converges in capacity to $\phi \in PSH(X,\omega)$ if*

$$\forall t > 0\ \ cap_\omega(|\phi_j - \phi| > t) \to 0 \text{ for } j \to \infty.$$

We refer to [K3], [GZ1] for the basic properties of the relative capacity and the notion of convergence with respect to it. Recently Hiep in [Hi] obtained the following characterization of convergence in capacity for uniformly bounded functions:

**Theorem 3.1.5.** *Let $\phi_j$, $\phi$ be uniformly bounded $\omega$-psh functions. The following are equivalent:*

(1) *$\phi_j$ converges to $\phi$ in capacity,*
(2) $\limsup_{j\to\infty} \phi_j \leq \phi$ *and* $\int_X (\phi_j - \phi)\,\omega_{\phi_j}^n \to 0$.

Below we state an important capacity estimate due to Kołodziej ([K3]):

**Theorem 3.1.6.** *Let $\phi, \psi \in PSH(X,\omega)$. If $\phi$ satisfies $0 \leqslant \phi \leqslant C$, then for $s < C+1$, we have*

$$Cap_\omega(\{\psi + 2s < \phi\}) \leqslant \left(\frac{C+1}{s}\right)^n \int_{\{\psi+s<\phi\}} (\omega + dd^c\,\psi)^n.$$

*Proof.* Define $E(s) := \{\psi + s < \phi\}$. Take any $\rho \in PSH(X, \omega)$ valued in $[-1, 0]$. Set $V = \{\psi < \frac{s}{C+1}\rho + (1 - \frac{s}{C+1})\phi - s\}$. Since $-s \leqslant \frac{s}{C+1}\rho - \frac{s}{C+1}\phi \leqslant 0$, we can easily deduce the following chain relation of sets:

$$E(2s) \subset V \subset E(s).$$

Then we obtain the following:

$$\begin{aligned}(\frac{s}{C+1})^n \int_{E(2s)} (\omega + \sqrt{-1}\partial\bar{\partial}\rho)^n &\leqslant \int_V (\frac{s}{C+1}\omega_\rho + (1 - \frac{s}{C+1})\omega_\phi)^n \\ &\leqslant \int_V {\omega_\psi}^n \leqslant \int_{E(s)} {\omega_\psi}^n,\end{aligned}$$

where we have used the relation of the sets above and then applied the comparison principle for the two functions appearing in the definition of the set $V$.

Finally we can conclude the result from the definition of $Cap_\omega$. □

Below we prove an analogous result with slightly modified parameters. We shall use this proposition later.

**Proposition 3.1.7.** *Let $\phi, \psi \in PSH(X, \omega)$ satisfy $0 \leqslant \phi \leqslant a$, $0 \leqslant \psi \leqslant a$. Then for every $m$, $n$, $t > 0$ we have the inequality*

$$Cap_\omega(\{\psi + (m+n)t < \phi\}) \leqslant (\frac{a+1}{nt})^n \int_{\{\psi + mt < \phi\}} (\omega + dd^c\psi)^n.$$

*Proof.* If $nt \geq a + 1$ then the set $\{\psi + (m+n)t < \phi\}$ is empty (because of the additional assumption on $\psi$) and the proposition holds trivially. If, in turn, $nt < a + 1$ then for any function $\rho \in PSH(X, \omega)$, $-1 \leq \rho \leq 0$ we have

$$0 \leq \frac{nt\phi}{1+a} - \frac{nt\rho}{a+1} \leq \frac{nta}{a+1} + \frac{nt}{a+1} = nt.$$

These inequalities give us the set inclusions

$$\{\psi + (m+n)t < \phi\} \subset \{\psi + mt < (1 - \frac{nt}{a+1})\phi + \frac{nt}{a+1}\rho\} \subset \{\psi + mt < \phi\}.$$

From now on the reasoning is entirely analogous to the one from Theorem 3.1.6. □

3.1.2. *Cegrell classes.* As in the flat case, it is desirable to enlarge the domain of definition of the complex Monge-Ampère operator. Thus one is led to the definition of Cegrell classes. Below we discuss the analogues of Cegrell classes in Kähler manifold setting. We refer to [GZ2] and [Di1] for more details.

For every $u \in PSH(X, \omega)$ $(\omega + dd^c\, max(u, -j))^n$ is a well defined probability measure. By [GZ2] the sequence of measures $\chi_{\{u > -j\}}(\omega + dd^c\, max(u, -j))^n$ is always increasing and one defines

$$\mathcal{E}(X, \omega) := \{u \in PSH(X, \omega) \mid \lim_{j \to \infty} \int_X \chi_{\{u > -j\}}(\omega + dd^c\, max(u, -j))^n = 1\}.$$

These functions are a priori unbounded, but the integral assumption ensures that the Monge-Ampère measure has no mass on $\{u = -\infty\}$. Then one defines

$$(\omega + dd^c\, u)^n := \lim_{j \to \infty} \chi_{\{u > -j\}}(\omega + dd^c\, max(u, -j))^n.$$

In particular Monge-Ampère measures of functions from $\mathcal{E}(X, \omega)$ do not charge pluripolar sets. We refer to [GZ2] for a discussion of that notion.

The class $\mathcal{E}^1(X,\omega)$, or more generally $\mathcal{E}^p(X,\omega), p>0$ is defined by

$$\mathcal{E}^p(X,\omega) := \{\, \phi \in \mathcal{E}(X,\omega) \ | \ \int_X |\phi|^p \omega_\phi^n < \infty \,\}\,.$$

Since $\omega$-psh functions are upper semicontiunuous, they are bounded from above, hence one usually considers only negative $\omega$-psh functions from $\mathcal{E}^p(X,\omega)$, which often comes in handy in technical details. Note that originally the classes $\mathcal{E}^p$ were defined (similarly to the Cegrell classes in the flat theory) by a sequence of bounded functions $\phi_j, \phi_j \searrow \phi$, such that $\sup_j \int_X |\phi_j|^p \omega_{\phi_j}^n < \infty$. The results from [GZ2, Di1] have shown that actually one can take just the sequence $\phi_j := \max(\phi, -j)$, hence both definitions are coherent.

One can also define "local" classes in an attempt similar to the one from [Bl4, Bl3]. We define the class $\mathcal{D}(X,\omega)$ by

$$\mathcal{D}(X,\omega) := \{\, \phi \in PSH(X,\omega) \mid \forall z \in X \ \exists U_z - \text{open},\ z \in U_z,\ \rho + \phi \in \mathcal{D}(U_z) \,\},$$

where $\rho$ is a local potential in $U_z$ for $\omega$ and $\mathcal{D}(U_z)$ is the maximal domain of definition of the Monge-Ampère operator in $U_z$, (see [Bl4, Bl3]).

Note however that the "local" and global definition yield different classes, as shown in [GZ2]. This is in sharp contrast with the "flat" theory.

Define also $\mathcal{D}^a(X,\omega)$ by

$$\mathcal{D}^a(X,\omega) := \{\, \phi \in \mathcal{D}(X,\omega) \ | \ \omega_\phi^n(A) = 0,\ \forall A \subset X,\ A - \text{pluripolar} \,\}\,.$$

It is known that $\mathcal{D}^a(X,\omega) \subset \mathcal{E}(X,\omega)$, while $\mathcal{D}(X,\omega) \nsubseteq \mathcal{E}(X,\omega) \nsubseteq \mathcal{D}(X,\omega)$.

Note that the terminology in the Cegrell classes, partially due to the mentioned differences in the "local" and "global" settings varies in the literature. In particular the class $\mathcal{D}(X,\omega)$ is denoted by $\mathcal{E}(X,\omega)$ in [HKH] or [Di2]. The class $\mathcal{E}(X,\omega)$ in turn differs in some aspects from the class $\mathcal{E}(\Omega)$ in the "flat" setting (for example a function in $\mathcal{E}(\Omega)$ may have a Monge-Ampère measure that charges points).

Let us state a result that we shall need late on - an inequality for mixed Monge-Ampère measures. This already follows form our discussion of Theorem 2.1.54, since the measures we consider do not charge pluripolar sets and the inequality in the cited theorem is local.

**Theorem 3.1.8.** *Let $u,\ v \in \mathcal{E}(X,\omega)$ be $\omega$-psh functions, $\mu$ be a positive measure that does not charge pluripolar sets and $f,\ g \in L^1(d\mu)$. If*

$$(\omega + dd^c\, u)^n \geq f d\mu,\ (\omega + dd^c\, v)^n \geq g d\mu$$

*as measures, then*

$$(\omega + dd^c\, u)^k \wedge (\omega + dd^c\, v)^{n-k} \geq f^{\frac{k}{n}} g^{\frac{n-k}{n}} d\mu,\ \forall k \in \{1, \cdots, n-1\}.$$

**Corollary 3.1.9.** *If $\phi,\ \psi \in \mathcal{E}(X,\omega)$ and $\omega_\phi^n = \omega_\psi^n$, then for every $t \in (0,1)$ we have*

$$\omega^n_{t\phi + (1-t)\psi} = \omega_\phi^n = \omega_\psi^n\,.$$

The next result we shall need is somewhat nonstandard, so although it is similar to the usual comparison principle, we sketch a proof.

**Theorem 3.1.10** ("partial" comparison principle)**.** *Suppose $T$ is a $(k,k)$ positive closed current on $X$ of the form $\omega_{\phi_1} \wedge \cdots \wedge \omega_{\phi_k}$, $\phi_j \in \mathcal{E}(X,\omega)$, where $0 \leq k \leq n-1$. Let furthermore $u,\ v \in \mathcal{E}(X,\omega)$. Then*

$$\int_{\{\, u<v \,\}} \omega_v^{n-k} \wedge T \leq \int_{\{\, u<v \,\}} \omega_u^{n-k} \wedge T.$$

*Proof.* Note that in the case $k = 0$ this is the standard comparison principle in $\mathcal{E}(X, \omega)$, which was shown in Theorem 1.5 in[GZ2], (historically the bounded case was first shown in [K3]). Note also that it is enough to get the statement for $n-k = 1$, since all the other cases can be done by iteration of the "$n - k = 1$" argument. So, we assume $n - k = 1$.

In the class $\mathcal{D}^a(X, \omega)$ the claimed result was proven in [HKH]. Our case is indeed not much different.

Basically one can repeat the argument from the Theorem 1.5 in [GZ2], provided one knows that for bounded $u,\ v$

$$\chi_{\{u<v\}}\omega_v \wedge T = \chi_{\{u<v\}}\omega_{\max(u,\ v)} \wedge T,$$

corresponding to equality (1) in the proof of Guedj and Zeriahi. Now one can proceed in the following way: define the canonical approximants for $\phi_1, \cdots, \phi_{n-1}$ by

$$\phi_s^{(j)} := max(\phi_s, -j).$$

Let $T^{(j)} := \omega_{\phi_1^{(j)}}, \wedge \cdots \wedge \omega_{\phi_{n-1}^{(j)}}$. All the functions appearing in the wedge products belong to $\mathcal{D}^a(X, \omega)$, hence [HKH] applies. So, for $u,\ v$ bounded we have

$$\int_{\{u<v\}} \omega_v \wedge T^{(j)} \leq \int_{\{u<v\}} \omega_u \wedge T^{(j)}.$$

By propreties of $\mathcal{E}(X, \omega)$ proven in [GZ2] for bounded $u,\ v$ when passing to the limit with $j \to \infty$ one gets

$$\int_{\{u<v\}} \omega_v \wedge T \leq \int_{\{u<v\}} \omega_u \wedge T.$$

Now the step from bounded $u,\ v$ to general $u,\ v$ can be done exactly as in Theorem 1.5 in [GZ2].

□

## 3.2. **The Monge-Ampère equation.**

3.2.1. *Calabi-Yau theorem and generalizations.* The Monge-Ampère operator appears in complex geometry in the formula for the Ricci curvature of a Kähler metric:

**Definition 3.2.1 (Ricci curvature).** *If* $\omega = \frac{i}{2}\sum_{k,j=1}^{n} g_{k\bar{j}} dz^k \wedge d\bar{z}^j$ *is a Kähler form (with associated metric* $(g_{k\bar{j}})$*) we define its Ricci curvature form* $Ric_\omega$ *by*

$$Ric_\omega := -i\partial\bar{\partial}log(det(g_{k\bar{j}})).$$

Observe that this definition a priori depends on the choice of the local coordinate system. However under any other coordinates the quantity $\omega^n$ will differ by the square of the modulus of the Jacobian of the (holomorphic) change of variables. Since $log(|F|^2)$ is a pluriharmonic function for $F$ holomorphic, it turns out that the so defined form is independent of the local coordinate system and hence defines a global $(1, 1)$-form.

Actually one can say more: the form $Ric_\omega$ belongs to the first Chern class $c_1(X)$ (see, for example [GH] or [T]). Due to the geometers' interest in the Ricci curvature it became important to study more thoroughly the possible elements of $c_1(X)$ which are "generated" as Ricci forms of some metric. This was the starting point of the Calabi conjecture:

*If we fix a (positive) cohomology class* $[\omega] \in H^{1,1}(X,\ \mathbb{R})$ *on a compact Kähler manifold* $(X, \omega)$ *and a form* $\theta \in c_1(X)$ *is it always possible to find a repersentative* $\eta \in [\omega]$ *which is positive (i.e. a Kähler) and its Ricci curvature satisifes* $Ric_\eta = \theta$*?*

This conjecture would for example imply that on any Kähler manifold $X$ with $c_1(X) = 0$ one can find Ricci-flat Kähler metric on (that is $\eta$, such that $Ric_\eta = 0$) within any Kähler class in $X$. This, in turn, implies a lot of geometrical and even topological information on $X$ (see, for example, [T] for details).

By the so called $\partial\bar{\partial}$ lemma (see [De2] or [T]) any form cohomologous to $\omega$ is of type $\omega + i\partial\bar{\partial}u$ for some real function $u$. Thus we get the two equations

$$\begin{aligned} Ric_{\omega+\partial\bar{\partial}u} &= -i\partial\bar{\partial}log((\omega + \partial\bar{\partial}u)^n) = \theta + i\partial\bar{\partial}f, \\ Ric_{\omega} &= -i\partial\bar{\partial}log((\omega)^n) = \theta \end{aligned}$$

(by an abuse of notation here we denote the determinant of $(g_{k\bar{j}})$ by $\omega^n$). By extracting the second one from the first one obtains

$$-i\partial\bar{\partial}log(\frac{(\omega + \partial\bar{\partial}u)^n}{\omega^n}) = i\partial\bar{\partial}f. \tag{3.1}$$

Since on a compact Kähler manifold there are no non constant pluriharmonic functions we obtain from 3.1 that

$$log(\frac{(\omega + \partial\bar{\partial}u)^n}{\omega^n}) = -f + c,$$

for some constant $c$. By rewriting the above one gets the Monge-Ampère equation

$$(\omega + \partial\bar{\partial}u)^n = e^{-f+c}\omega^n.$$

Thus the solution of the Calabi conjecture boils down to solving a Monge-Ampère equation. This was done by Yau in his seminal paper [Y]:

**Theorem 3.2.2** (The Calabi-Yau theorem)**.** *The Monge-Ampère equation arising from 3.1 has a smooth solution whenever the function $f$ is smooth.*

The proof of the Calabi-Yau theorem is beyond the scope of this note. We shall only briefly sketch the main ideas.

The proof uses the so called *continuity* method. One starts with data on the right hand side for which the equation is solvable (for example, if $f \equiv 1$ then clearly $\phi = 0$ is a solution), and then tries to perturb the data via continuous path towards the fixed Dirichlet problem (one may take the "data" path $t \to f(t) = tf + (1-t)$). The goal is to show that the equation is solvable for all $t \in [0, 1]$. It is enough to prove that the set of those $t \in [0, 1]$ for which $(\omega + \partial\bar{\partial}u_t)^n = f(t)\omega^n$ is solvable is both open and closed. For the openness one can argue by a kind of implicit function theorem in the Banach space $\mathcal{C}^\infty(X)$ (see [T]) to show that whenever the equation is solvable for $t_0$ it is also solvable for times close enough to $t_0$.

The hard part is the closedness. It is enough to show that having a solution for times $t_i$ with $\lim_{i\to\infty} t_i = t_0$ one can also solve the problem for $t_0$. To show this one can use Arcela-Ascoli theorem for the sequence $u_{t_i}$ (viewed as a sequence in the Banach space $\mathcal{C}^{2,\alpha}(X)$) provided one can obtain a priori $C^0$, $C^1$, $C^2$ and $C^{2,\alpha}$ estimates for the solutions. This is the technical heart of the Yau's proof in [Y].

Historically the $C^3$ estimates (instead of $C^{2,\alpha}$) were used and these were known for some time (we refer to [Si1] for more details on the history of the problem). The $C^2$ estimate turned out to be independent of the $C^1$ one (which is a very rare situation in nonlinear PDE theory) and thus the proof is reduced to the $C^0$ and $C^2$ estimates. While the proof of $C^2$ a priori estimate can be handled by PDE techniques, the $C^0$ estimate (surprisingly) turned out to be the most difficult one. In his original proof Yau [Y] used the Moser iteration technique coupled with reverse Hölder inequalities and Sobolev type estimates to obtain the result.

The methods used were rather restrictive and soon new geometric problems (connected, for example, with the limiting behaviour of the equation if we let the Kähler form to vary) became intractable.

At this stage Kołodziej, using pluripotential methods ([K2]) proved the $C^0$ estimate for the weak solutions of equation

$$(3.2)\qquad (\omega + dd^c\,\phi)^n = F\omega^n,\ \text{where } F \geq 0,\ F \in L^p(\omega^n),\ p > 1,\ \int_X F\omega^n = \int_X \omega^n.$$

The main advantage of this proof over Yau's is that the estimate is independent neither on $inf_X F$ nor on the smoothness of this function.

Below we sketch the main ideas of Kołodziej's proof. It hinges on several facts that allow to reduce the a priori estimate to a local problem from flat theory.

First, if $u$ is a negative $\omega$-psh function satisfying $\sup_X u = 0$, then $\int_X u\omega$ is always finite and can be bounded from below by $-c_1$ for some constant $c_1$ depending only on $(X, \omega)$ (in the sequel we enumerate the constants $c_i$ in order to distinguish them). Thus in any coordinate chart $U$ one has $\int_U u\omega^n > -c_1$. So, for any finite open cover (that is, a finite collection of charts $V_i,\ i = 1, \cdots, N$, such that $\cup_{i=1}^N V_i = X$) one has

$$(3.3)\qquad sup_{V_i} u > -c_2,$$

with a constant $c_2$ dependent on $X$ and the covering but independent of $u$.

The second fact we need is that for any point $z \in X$ we can find in a neigbourhood a potential $\eta$ for the Kähler metric which has a (strict) minimum at $z$. This is done in the following way:

In local coordinates in a ball $B''$ centered at $z$ any potential $\rho$ of the form $\omega$ is a strictly plurisubharmonic smooth function and can be expanded as

$$\rho(z+h) = \rho(z) + 2\Re\Big(\sum_{j=1}^n a_j h_j + \sum_{j,k=1}^n b_{jk} h_j h_k\Big) + \sum_{j,k=1}^n c_{j\bar{k}} h_j \bar{h}_k + o(|h|^2)$$
$$= \Re P(h) + H(h) + o(|h|^2),$$

where $P$ is a complex polynomial in $h$ and $H$ is the complex Hessian at $z$.

Proceeding exactly as in [K2] (Lemma 2.3.1) $\eta := \rho - \Re P(\cdot - z)$ is also a local potential for $\omega$, with the additional property that $\eta$ has a strict local miniumum at $z$ (we use at this point that $H$ is strictly positive definite). This means that for smaller balls $B$ and $B'$ satisfying $B \Subset B' \Subset B''$ if $B$ is sufficiently small one has $\inf_{\partial B'} \eta > \sup_B \eta + c_3$ for some positive constant $c_3 > 0$ dependent of the positivity of $\omega$ and the modulus of continuity of $\eta$.

So, by compactness one may take finite collection of such triples of balls $B_z \Subset B'_z \Subset B''_z$, such that $B_z$ form an open covering.

We fix a point $c \in X$ where the function $u$ obtains a minimum (recall that we prove an a priori inequality, so we assume $u$ is continuous - in such a case the minimum is attained). We choose one of the coordinate balls $B$ (we drop, for the sake of notational ease, the subscript $z$ in the sequel) such that $c \in B$. Fix also a ball $B^*$ such that $B' \Subset B^* \Subset B''$. Then $sup_{\bar{B}} u = u(x) > -c_2$ for some $x \in \bar{B} \Subset B'$, and by construction the function $v := u + \eta$ satisfies the following:

$$(3.4)\qquad v \text{ is plurisubharmonic in } B'',$$

$$(3.5)\qquad v(x) > -c_4 \text{ for some } c_4 > 0 \text{ depending only on } X,$$

$$(3.6)\qquad v(c) \leq inf_{\partial B'} v - c_5 > 0 \text{ for some } c_5 \text{ depending only on } X.$$

Thus the proof is reduced to the following local result (below we make the notation suggestive to make the correspondences between notions clear):

*Let* $B^{"}$ *be a strictly pseudoconvex domain in* $\mathbb{C}^n$. *Let also the negative plurisubharmonic function* $v$ *satisfies* $(dd^c v)^n = f dV$ *for some* $f \neq 0,\ f \in L^p(dV),\ p > 1$. *Assume moreover that for some point* $x$ *we have* $v(x) > -c_4$ *and the sets* $U(s) := \{\, z|\ v(z) < -s\,\} \cap B'$ *are nonempty and relatively compact in the domain* $B' \Subset B^* \Subset B^{"}$ *for any* $s$ *in an interval of lenght at least* $c_5$. *Then*

$$-inf_{B'} v \leq C(c_4,\ c_5,\ B',\ B^*,\ B^{"},\ p,\ ||f||_p),$$

*for some constant* $C$ *dependent only on the mentioned quantities.*

This bound gives us uniform esitmate of $v$ and hence also of $u$ on $X$. For the (technically involved) proof of this local fact we refer to [K2], Lemma 2.3.1. We only mention that the proof depends crucially on the following Lemma:

**Lemma 3.2.3** ([K2])**.** *If* $f \geq 0,\ f \in L^p(dV, B^{"}),\ p > 1$ *then there exists an increasing function* $Q : (0,\ \infty) \to (0,\ \infty)$ *satisfying*

(1) $\int_1^\infty (yQ^{1/n}(y))^{-1} dy < +\infty$,

(2) *there exists a constant* $A$, *dependent only on* $B^{"}$, $n$ *and* $Q$, *such that for each compact* $K \subset B^{"}$

$$\int_K f dV \leq A cap(K, B^{"})][Q(cap(K, B^{"})^{-1/n})]^{-1}.$$

This estimate, roughly speaking, says that the Monge-Ampère measure of $v$ decays for small compacts $K$ a bit faster than the capacity. The rate of the decay is measured by the function $Q$. It was in fact proved in [K2] that one may take $Q(t) = c_m t^m$ for any positive exponent $m$ (the constant $c_m$ increases with $m$). For our later reference we define an auxiliary function

$$\kappa(s) := A^{1/n}[\int_{s^{-1/n}}^\infty y^{-1} Q^{-1/n}(y) dy + Q^{-1/n}(s^{-1/n})]. \tag{3.7}$$

It is easy to compute that with the choice of $Q$ as above one gets $\kappa(t) = c_{m,n} t^{\frac{m}{n^2}}$.

After [K2] several other proofs of the $L^\infty$ estimate have appeared (see, for example [K3], [K4] and [EGZ]). In all these newer proofs instead of localizing the estimate one works globally on the manifold $X$. To this end instead of the capacity $cap(K, B^{"})$ one has to work with the relative capacity $cap_\omega$. It can be proved however that the analogue of Lemma 3.2.3 also holds. We have chosen to present the original argument since we shall need some of the introduced concepts later on.

The next step is of course the quest for a better regularity of the weak solutions. In [K2] it was shown that any solution of (3.2) is in fact continuous. We postpone the proof of this fact to the next subsection where this will be treated in a more general situation.

3.2.2. *The big form case.* When analyzing various problems in geometry, one often has to let the Kähler class $\omega$ vary. Especially interesting are the limiting situations, i.e. the cases when a family a Kähler forms tends to a form in a non-Kähler class. A typical example is the limiting behavior at infinity of the Kähler-Ricci flow when the canonical divisor is big and nef. One of the situations that may appear involves the so called *big* forms:

**Definition 3.2.4 (Big form).** *A closed semi-positive* $(1,1)$*-form* $\omega$ *on a compact Kähler manifold* $X$ *is called big if*

$$\int_X \omega^n > 0.$$

The difference with the kählerness condition is the assumption of semi-postivity instead of strict positivity. Let us give an example in order to emphasize the differences:

**Example 3.2.5.** *Let $X$ and $Y$ be complex manifolds and let $\pi : X \to Y$ be a holomorphic map which is generically finite-to-one i.e. the preimage of a generic point form $Y$ is finite. If $\omega$ is a Kähler form on $Y$ then $\pi^*\omega$ is a semi-positive form on $X$ satisfying additionally $\int_X \pi^*\omega^n > 0$, but it may not be Kähler. A local example of this can be obtained by taking $\pi : \mathbb{C}^2 \ni (z_1, z_2) \to (z_1, z_2^2) \in \mathbb{C}^2$. Then $\pi^* \, dd^c \, ||z||^2$ is not strictly positive on the line $\{ z_2 = 0 \}$.*

Note that analogously to the Kähler case one can define $\omega$-psh functions in the big setting. It is interesting to consider the corresponding Monge-Ampère equation in the big setting:

$$(\omega + dd^c \phi)^n = F\omega^n, \ \phi \in PSH(X,\omega), \ F \in L^p(\omega^n), \ p > 1. \tag{3.8}$$

In [EGZ] and [Z] (see also [BGZ]) it was shown that analgously to the Kähler case one has the $L^\infty$ estimate:

**Theorem 3.2.6.** *Let $\omega$ be a big form on a compact Kähler manifold and let $\phi \in PSH(X,\omega)$ solve the equation*

$$(\omega + dd^c \phi)^n = F\omega^n, \ sup_X\phi = 0, \ F \in L^p(\omega^n), \ p > 1.$$

*Then there exists a constant $C$ depending only on $X$, $\omega$, $p$, $||F||_p$ such that*

$$||\phi||_\infty \leq C.$$

In the Kähler case, as we have already mentioned, the continuity of such a solution was shown in [K2]. Therefore it is natural to expect also a continuity result in the big form setting. The problem turned out to be a highly nontrivial one and is still open except in a special situation which we shall now describe.

**Theorem 3.2.7.** *Let $(X, \ \omega_X), (Z, \ \omega_Z)$ be two Kähler manifolds and $F : X \to Z$ be a holomorphic mapping with the property that the image of $X$ is locally birational to $X$. Then $\omega := F^*\omega_Z$ is a big form on $X$ and the solution on the Monge-Ampère equation (3.8) is continuous.*

**Remark 3.2.8.** *In geometrical applications the manifold $Z$ is usually the projective space $\mathbb{P}^N$ for some $N \in \mathbb{N}$. Therefore we shall give the proof in the case $(Z, \ \omega_Z)$ is simply the projective space equipped with the Fubini-Study metric, but the general case goes the same way. Arguments used, however, heavily rely on* [K2] *and at some places we just follow it line by line.*

This theorem was obtained by Zhang in [Z]. The proof there is hovewer a bit too sketchy and the details were furnished in [DZ]. Here we follow the exposition given in the latter paper.

First of all we recall the geometrical background. Let $X$ be the base closed Kähler manifold we work on, and $F : X \to \mathbb{P}^N$ be a map with the property that the image $F(X)$ has the same dimension and $F$ is itself locally birational i.e. for every small enough neighbourhood $U$ of any point on $F(X)$, each component of $F^{-1}(U)$ is birational to $U$. A typical global example of this situation is obtained as follows: if $X$ carries a big line bundle $L$, the linear series corresponding to $L^n$ generate (for sufficiently big $n \in \mathbb{N}$) a birational morphism into $\mathbb{P}^N$ with the claimed properties. Note hovewer that local and global birationality are different notions (see the example below) and if one has to deal with the global birationality one has to impose some additional assumptions for the argument to go through.

Consider now $Y := F(X)$. By the Proper Mapping Theorem $Y$ is a (singular in general) subvariety in $\mathbb{P}^N$. It is also clear that $Y$ is irreducible and locally irreducible variety (the latter follows from the local birationality). Recall that an upper semicontinuous function $u$ on a singular variety $D$ is called weakly plurisubharmonic if for every holomorphic disc $f : \Delta \to D$ the function $u \circ f$ is a subharmonic function (see [FN]). In that paper it is proved (in fact in a much more general situation of Stein spaces) that any such function $u$ can be extended locally to the ambient space to a classical plurisubharmonic function i.e. for every $x \in Y$ there exists a small Euclidean ball $B$ in $\mathbb{P}^N$, centered at $x$ and a function $v \in PSH(B)$, such that $v|_{B\cap Y} = u$.

Suppose now that $\phi$ is a positive discontinuous solution of the Monge-Ampere equation in question and let $d := \sup(\phi - \phi_*) > 0$, where $\phi_*$ denotes the lower semicontinuous regularization of $\phi$. Note that the supremum is attained, and if $E$ is the closed set $\{\phi - \phi_* = d\}$, there exists a point $x_0$ such that $\phi(x_0) = \min_E \phi$. Positivity is a technical assumption that can always be achieved by adding appropriate constant since we already know that $\phi$ is bounded.

By assumption there exist analytic sets $Z \subset X$ and $W \subset Y = F(X)$ such that $F|_{X\setminus Z} \to Y \setminus W$ is a biholomorphism and moreover $S := \{\omega^n = 0\} \subset Z$. Note that in the general case of a big form $S$ need not be contained in an analytic set - it may well happen that $S$ is open in $X$.

Two possibilities might take place

1. $x_0 \in X \setminus S$. In this case $\omega$ is strictly positive in a small ball centered at $x_0$ and repeating the argument from Section 2.4 in [K2] we obtain a contradiction.
2. $x_0 \in S$. Then we shall produce a domain $V$ (not contained in a chart in general) and a potential $\theta$ of $\omega$ in $V$ with the property that $\inf_{\partial V} \theta > \theta(x_0) + b$, where $b$ is a positive constant.

Consider $F(x_0) = z$ and a neighbourhood $U$ of $z$ in $\mathbb{P}^N$, such that its preimages are birational to it. Choose the one $x_0$ sits in. For the rest of the argument we restrict ourselves to $F|_{F^{-1}(U)\ni x_0} \to U$. Consider the pushforward function

$$F_*\phi(z) := \begin{cases} \phi(w), \quad if\ z \in Y \setminus W, w \in X \setminus Z,\ F(w) = z \\ \limsup_{X\setminus Z\ni\zeta\to z} F_*\phi(\zeta)\ if z \in W \end{cases}$$

and a local potential $\eta$ for the Kähler form on $U$.

Claim: $\eta + F_*\phi$ is weakly subharmonic on $Y$.

Proof: Weak subharmonicity is a local property, hence it is enough to check it in a neighbourhood of any point on $Y$. For regular points of $Y$ this is evident. However at singularities of $Y$ one might a priori run into trouble as the example of a double point shows. Indeed, consider the following (classical) local example:

Let

$$F : \mathbb{C} \ni t \to (t^2 - 1, t(t^2 - 1)) \in \mathbb{C}^2$$

The image $F(\mathbb{C})$ sits in the variety $\{(z_1, z_2) \in \mathbb{C}^2 | z_1^2 + z_1^3 = z_2^3\}$. Observe that $F$ is a bijection onto its image, except for the points 1 and $-1$ being mapped to $(0, 0)$. But then it is clear that the pushforward of a subharmonic function $v$ on $\mathbb{C}$ cannot be weakly subharmonic on the image if $v(1) \neq v(-1)$. Note that $F$ is not locally birational though.

Observe that local birationality forces the analytic set $Y$ to be locally irreducible. Then there is a classical theorem (see [De1], Theorem 1.7) stating that on a locally irreducible variety $Y$ any locally bounded plurisubharmonic function $w$ defined on $\operatorname{Reg} Y$ (the regular part of $Y$) can be extended via limsup technique $v(z) := \limsup_{\zeta \to z,\ \zeta \in \mathrm{Reg}\, Y} v(\zeta)$ to a weak plurisubharmonic function. Moreover, it follows from the proof that for any $s \in Y$ and any birational modification $G : Y^{'} \to Y$ of $Y$ the pulled back function $G^* w$ is constant on the fiber $G^{-1}(s)$.

Now, if $\omega_{\mathbb{P}^N}$ is the Kähler metric which defines $\omega$ (i.e. $\omega = F^*\omega_{\mathbb{P}^N}$), fix $\rho$ - a local potential of $\omega_{\mathbb{P}^N}$ in a neighborhood of $z$ in $\mathbb{P}^n$ (we can assume without loss of generality that this neighborhood coincides with $U$). We modify $\rho$ exactly as in the proof of $L^\infty$ estimate in the previous subsection. So, we can assume that $\rho$ has at $z$ strict local minimum.

By the classical Fornaess-Narasimhan theorem ([FN], Theorem 5.3.1) we find a small euclidean ball $B^{'}$ in $\mathbb{P}^n$ centered at $z$ and a function $\psi \in PSH(B^{'})$, such that $\psi|_{Y \cap B^{'}} = \eta + \phi$. On a neighbourhood of a slightly smaller ball $B$ (everything is contained in $B^{'}$ and $U$) $\psi$ can be approximated by a sequence of smooth plurisubharmonic functions $\psi_j$ decreasing towards it. Again (decreasing a bit $B$ if necessary) one can get $\inf_{\partial B} \eta > \eta(z) + b$ for some positive constant $b$. Now we pull back the ball and the regularizations: let $V := F^{-1}(B^{'} \cap Y)$ and $u_j := \psi_j(F(w))$ ( $u_j$ are assumed to be defined only on small neighbourhood of $V$). Of course these are continuous plurisubharmonic functions on $V$ which decrease towards $u := \eta \circ F + F^*(F_*\phi) = \eta \circ F + \phi$ (the equality is due to the fact that $\phi$ has to be constant on each fiber). Note that $V$ need not be an Euclidean domain anymore (i.e. it need not be contained in a coordinate chart), nevertheless $\eta \circ F$ is a global potential of $\omega$ on this set. This is the essential difference between this special situation and the general case. Next we state a lemma which is essentially contained in [K2] (Section 2.4). We include the proof for the sake of completeness.

**Lemma 3.2.9.** *There exist $a_0 > 0,\ t > 1$ such that the sets*

$$W(j, c) := \{tu + d - a_0 + c < u_j\}$$

*are non-empty and relatively compact in $V$ for every constant $c$ contained in an interval independent of $j > j_0$.*

*Proof.* Note that $E(0) := \{u - u_* = d\} \cap \overline{V} = E \cap \overline{V}$, since the potential is continuous. Also the sets $E(a) := E := \{u - u_* \geq d - a\} \cap \overline{V}$ are closed and decrease towards $E(0)$. Hence if $c(a) := \phi(x_0) - \min_{E_a} \phi$ we have that $\limsup_{a \to 0^+} c(a) \leq 0$, for othrwise we would get a contradiction with the definition of $d$. Hence

$$c(a) < \frac{1}{3} b$$

for $0 < a < a_0 < min(\frac{1}{3}b, d)$. Let $A := u(x_0)$. Note that $A > d$, since the potential is greater than 0 at $x_0$, and $\phi$ as a globally positive function has to be greater than $d$ at $x_0$. One can choose $t > 1$, such that it satisfies

$$(t-1)(A-d) < a_0 < (t-1)(A - d + \frac{2}{3} b).$$

Now, if $y \in \partial V \cap E(a_0)$ one gets

$$u_*(y) \geq \eta(F(x_0)) + b + F^* F_* \phi(x_0) \geq A - d + \frac{2}{3} b.$$

Hence $u(y) \leq u_*(y) + d < t u_*(y) + d - a_0$. Note that this inequality extends to a neighborhood of $\partial V \cap E(a_0)$. Taking another neighborhood relatively compact in the first one and applying Hartogs type argument one obtains

$$u_j < tu(y) + d - a_0, \quad \forall j > j_1.$$

For the rest part of $\partial V$ the same inequality holds if we take big enough $j_1$ and the proof is even simpler, since $u - u_*$ is less than $d - a_0$ there. This proves the relative compactness on $W(j, c)$ in $V$.

Note that from the left inequality defining $t$ one gets $(t-1)u_*(x_0) < a_0$, hence

$$tu_*(x_0) < u(x_0) - d - a_1 + a_0 < u_j(x_0) - d - a_1 + a_0$$

for some constant $a_1 > 0$. This implies that the sets $W(j, c),\ c \in (0, a_1)$ contain some points near $x_0$, hence they are non empty. □

Now, by Lemma 2.3.1 from [K2] (one can verify that despite the fact that $V$ can be non Euclidean the argument still goes through) one can bound the capacity $cap(W(j, a_1), V)$ from below by an uniform positive constant. On the other hand $W(j, a_1) \subset \{u + (d - a_0 + a_1) < u_j\}$ and this contradicts the fact that the decreasing sequence $u_j$ has to converge towards u in capacity. This proves that $d = 0$, hence $\phi$ is continuous.

**Remark 3.2.10.** *As we have seen the argument cannot be applied in the case of a (globally) birational map. In this case some additional assumptions are needed to assure that the pushforward is plurisubharmonic. A satisfactory additional assumption is that the fibers in the preimage are connected. Then the function has to be constant on any nontrivial connected fiber and this is enough to push it forward onto the image.*

3.2.3. *The Monge-Ampère equation in Cegrell classes - existence.* The existence results for the Dirichlet problem in Cegrell classes were obtained by Guedj and Zeriahi in [GZ2]. Their proof is modelled on Cegrell's argument in the flat case [Ce2], [Ce3]. Let us state Guedj and Zeriahi's theorem:

**Theorem 3.2.11.** *Suppose $\mu$ is aprobability measure that does not charge pluripolar set. Then there exists $\phi \in \mathcal{E}(X, \omega)$ such that*

$$(\omega + dd^c\phi)^n = \mu.$$

Below we sketch the main ideas of the proof of this important result (see [GZ2] for the details).

The proof can be divided into two independent parts.

In the first part we prove that a probabilistic measure $\mu$ has a potential in $\mathcal{E}^1(X, \omega)$ if $\mathcal{E}^1(X, \omega) \subset L^1(\mu)$ (the converse implication also holds). To show this the measure $\mu$ is approximated (in the weak sense) by a special sequence of smooth strictly positive probabilistic measures $\mu_j$. This was done in [GZ2] by convoluting $\mu$ locally with smooth kernels and adding a small multiple of the volume form to obtain strict positivity. For each of the approximants $\mu_j$ one can find (by the Calabi-Yau theorem) a smooth potential $\phi_j$ normalized by, say, $sup_X \phi_j = -1$. Taking subsequence if necessary we find $\phi \in PSH(X, \omega)$ such that $\phi_j \to \phi$ in $L^1(\omega^n)$ and $sup_X \phi = -1$. The integrability assumption guarantees that in fact $\phi \in \mathcal{E}^1(X, \omega)$.

The most delicate point of the proof is to show that

$$lim_{j\to\infty} \int_X |\phi_j - \phi| d\mu_j = 0. \tag{3.9}$$

For this the Authors in [GZ2] rely heavily on the special choice of the approximating sequence. Having proven the fact above, we note that it is very similar to Hiep's criterion for convergence in capacity in the bounded case (Theorem 3.1.5), and indeed (3.9) is sufficient to show that

$$\omega_\phi^n = \mu,$$

which finishes the sketch of the first part of the proof.

The second ingredient is to solve the Dirichlet problem for a general probabilistic measure vanishing on pluripolar sets. For such a measure $\mu$, by applying Cegrell decomposition type argument (see Lemma 4.5 in [GZ2]) we find a function $u \in PSH(X,\omega) \cap L^{\infty}(X,\omega)$ and a function $f \geq 0$, $f \in L^1(\omega_u^n)$ such that $\mu = f\omega_u^n$. Then the sequence of measures $\mu_j := c_j min(f,j)\omega_u^n$ ($c_j \geq 1$ is a normalization constant, so that $\mu_j$ is a probability measure) approximates $\mu$. Note, that it is no loss of generality to assume $c_j \leq 2$, since this holds for $j$ big enough. Applying the first part one can find a potential $\psi_j \in \mathcal{E}^1(X,\omega)$, $sup_X\psi_j = -1$ for each of the measures $\mu_j$ (it is easy to show that not only $\mathcal{E}^1(X,\omega)$ but all $PSH(X,\omega)$ belongs to $L^1(\mu_j)$). Again, by choosing subsequence if necessary, we find $\psi \in PSH(X,\omega)$, $sup_X\psi = -1$, such that $\lim_{j\to\infty}\psi_j = \psi$. In fact one can extract a bit more information about $\psi$ (see the proof of Theorem 4.6 in [GZ2]) which turns out to be sufficient ot show that $\psi \in \mathcal{E}(X,\omega)$ and $\omega_\psi^n = f\omega_u^n = \mu$.

3.2.4. *The Monge-Ampère equation in Cegrell classes - uniqueness.* Since adding a constant does not influence the Monge-Ampère measure of the given function, the problem of classification of all possible solutions of the problem

$$(\omega + dd^c u)^n = d\mu,$$

is usually posed with a linear normalization condition $sup_X u = 0$ or $\int_X u\omega^n = 0$. But then, quite contrary to the corresponding problem in the "flat case", the problem turns out to be unexpectedly difficult.

Before we start any considerations we show an example that shows the failure of uniqueness in general:

**Example 3.2.12** ([Bl5]). *Let $X = \mathbb{P}^n, \omega = \omega_{FS}$ be the Fubini-Study Kähler form. Let*

$$\phi := \log\left(\frac{\sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}}{\sqrt{|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2}}\right),\ \psi := \max_{\{k=1,\cdots,n\}} \log\left(\frac{|z_k|}{\sqrt{|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2}}\right),$$

*(here $z = [z_0, \cdots, z_n]$ are the homogeneous coordinates in $\mathbb{P}^n$).*

*Then obviously $\phi - \psi$ is not constant, both functions have logarithmic pole at the point $c = [1 : 0, \cdots, : 0]$, and a calculation shows that*

$$(\omega + dd^c\phi)^n = (\omega + dd^c\psi)^n = \delta_c,$$

*where $\delta_c$ is the Dirac delta measure at the point $c$.*

Below we make some historical notes of positive results regarding the problem of uniqueness of normalized solutions:

The first result in this direction is due to E. Calabi ([Ca]). He proved that if $\phi, \psi$ are smooth and $\omega_\phi$, $\omega_\psi$ are Kähler forms (i.e. strictly positive) then uniqueness does hold. These are natural assumptions from geometer's perspective and the proof is quite easy in this case. However both smoothness and strict positivity are crucial in this approach, hence it gives no insight what to do in general.

The next step was done by Bedford and Taylor [BT5] who proved uniqueness for bounded $\phi, \psi$ provided the underlying manifold is $\mathbb{P}^n$. Their main idea was to control the $L^2$ norm of the gradient of the difference of $\phi$ and $\psi$.

Using different technique Kołodziej [K3] proved uniqueness for bounded functions on arbitrary compact Kähler manifold modulo additional mild assumptions on the measure $\mu$.

The "bounded" case was finally done by Błocki [Bl5]. The proof has some common points with the one in [BT5], but is much easier and transparent. Furthermore the proof gives some stability results showing that when one perturbs the measure on the right hand side slightly the normalized solution is in a way close to the original one.

In the Cegrell classes setting Guedj and Zeriahi in [GZ2] observed that Błocki's argument, with suitable modifications, can be carried over to prove uniqueness in the class $\mathcal{E}^1(X,\omega)$. Recently Demailly and Pali [DP] proved uniqueness in the same class for big forms. The most general result so far was proven very recently by Błocki (see [Bl6]) who proved that uniqueness does hold in the class $\mathcal{E}^{1-\frac{1}{2^{n-1}}}(X,\omega),\ n = dimX$.

Recall that the picture in the flat theory is much clearer due to a result of Cegrell who proved in [Ce3] that one can prove uniqueness provided the measure $\mu$ does not charge pluripolar sets. The proof however relies heavily on tools that are not available in the Kähler manifold setting. Nevertheless it is natural to expect that uniqueness in $\mathcal{E}(X,\omega)$ should also hold (observe that the functions from the example above are not in $\mathcal{E}(X,\omega)$).

Below we present the proof of uniqueness in $\mathcal{E}(X,\omega)$ taken from [Di3]. The proof hinges on different ideas than those in papers cited above, since functions in $\mathcal{E}(X,\omega)$ need not have bounded $L^2$ gradient and so the methods from [Bl5] and [GZ2] are not applicable (see however [Bl6] for some ideas in this vein). Instead we make use of tools developed in the former chapters.

**Theorem 3.2.13.** *Let $\phi,\ \psi \in \mathcal{E}(X,\omega)$ be such that $\omega_\phi^n = \omega_\psi^n$. Then $\phi - \psi$ is constant.*

*Proof.* Suppose on contrary that $\phi - \psi \neq const$. We will show that this leads to contradiction.

Consider first the level sets

$$A_t := \{\, \phi - \psi = t \,\},\ t \in \mathbb{R} \cup \{\, +\infty \,\} \cup \{\, -\infty \,\}.$$

These are all Borel sets which are closed in the plurifine topology. The main ingredient of the proof is to show that the whole mass of $\mu := \omega_\phi^n = \omega_\psi^n$ is concentrated on exactly **one** of the sets $A_t$. To achieve this, recall first that the measure charges at most countably many of the sets $A_t$ and does not charge neither $A_{+\infty}$ nor $A_{-\infty}$ (the first claim is proved in [GZ2] Corollary 1.10, and the second follow from the fact that both $A_{+\infty}$ and $A_{-\infty}$ are pluripolar).

We shall prove that $\mu$ charges precisely one of the sets $A_t$. Suppose the contrary. Then we can find $t_0 \in \mathbb{R}$ and a constant $1/2 < q < 1$ such that:

(1) $\int_{A_{t_0}} d\mu = 0$,
(2) $\int_{\{\,\phi<\psi+t_0\,\}} d\mu < q$,
(3) $\int_{\{\,\phi>\psi+t_0\,\}} d\mu < q$.

Indeed, one can find $t_1 \in \mathbb{R}$ such that

$$0 < \int_{\{\,\phi<\psi+t_1\,\}} d\mu < 1,$$

(for othervise the whole mass is concentrated on one level set). Now, if $\int_{A_{t_1}} d\mu = 0$, we take $t_0 := t_1,\ q = \max\{\, \int_{\{\,\phi<\psi+t_1\,\}} d\mu,\ 1 - \int_{\{\,\phi<\psi+t_1\,\}} d\mu \,\} + \epsilon$, with $\epsilon > 0$ so small that still $q < 1$. If $A_{t_1}$ is charged, then for almost every $t < t_1$ the set $A_t$ is not charged and by monotone convergence one can take $t_2 < t_1$ close enough to $t_1$ such that both $A_{t_2}$ is massless and still $0 < \int_{\{\,\phi<\psi+t_1\,\}} d\mu < 1$. Take $t_0 := t_2,\ q$ defined as before and we get the desired properties.

Since adding a constant to $\phi$ or $\psi$ is harmless for our discussion we assume from now on that $t_0 = 0$.

Consider the new measure

$$\widehat{\mu} := \begin{cases} (1/q)\mu, & on\,\{\,\phi < \psi\,\} \\ c\mu, & on\,\{\,\phi \geq \psi\,\}, \end{cases}$$

where $c$ is a nonnegative normalization constant so that $\widehat{\mu}$ is a probability measure (note that this is possible, since, by assumption, $\mu$ charges the set $\{\,\phi \geq \psi\,\}$).

Of course $\widehat{\mu}$ does not charge pluripolar sets either (and is also a Borel measure since the set $\{\,\phi \geq \psi\,\}$ is Borel). By [GZ2] we can solve the Monge-Ampère equation

$$\omega_\rho^n = \widehat{\mu},\ \rho \in \mathcal{E}(X,\omega),\ sup_X \rho = 0.$$

Note that at this moment we do not know if $\rho$ is uniquely defined: we just choose one solution.

In such a case we have a set inclusion

$$U_t := \{(1-t)\phi < (1-t)\psi + t\rho\,\} \subset \{\,\phi < \psi\,\}$$

for every $t \in (0,1)$. Hence on $U_t$ we have

$${\omega_\phi}^{n-1} \wedge \omega_{(1-t)\psi+t\rho} = (1-t)\mu + t\,\omega_\phi^{n\text{-}1} \wedge \omega_\rho \geq (1 + ((1/q)^{1/n} - 1)t)\,\omega_\phi^n,$$

where we have made use of Theorem 3.1.8 and Corollary 3.1.9.

So, by the comparison principle

$$(1 + ((1/q)^{1/n} - 1)t)\int_{U_t} \omega_\phi^n \leq \int_{U_t} \omega_\phi^{n\text{-}1} \wedge \omega_{(1-t)\psi+t\rho} \leq$$

$$\leq \int_{U_t} \omega_\phi^{n\text{-}1} \wedge \omega_{(1-t)\phi+t0} = (1-t)\int_{U_t} \omega_\phi^n + t\int_{U_t} \omega_\phi^{n\text{-}1} \wedge \omega.$$

Rearranging terms we obtain

$$(1/q)^{1/n}\int_{U_t} \omega_\phi^n \leq \int_{U_t} \omega_\phi^{n\text{-}1} \wedge \omega. \tag{3.10}$$

Note that exchanging $\omega_\phi^{n\text{-}1}$ with $\omega_\psi^{n\text{-}1}$ in the argument above gives

$$(1/q)^{1/n}\int_{U_t} \omega_\psi^n \leq \int_{U_t} \omega_\psi^{n\text{-}1} \wedge \omega. \tag{3.11}$$

(again we make use of Corollary 3.1.9 here). Now let $t \searrow 0$. The sets $U_t$ form an increasing sequence and $U_t \nearrow \{\,\phi < \psi\,\} \setminus \{\,\rho = -\infty\,\}$. But both measures $\omega_\phi^n$ and $\omega_\phi^{n\text{-}1} \wedge \omega$ do not charge pluripolar sets, hence we obtain

$$(1/q)^{1/n}\int_{\{\,\phi<\psi\,\}} \omega_\phi^n \leq \int_{\{\,\phi<\psi\,\}} \omega_\phi^{n\text{-}1} \wedge \omega. \tag{3.12}$$

One can do this reasoning also on the set $\{\,\phi > \psi\,\}$. Namely we find a measure defined like $\widehat{\mu}$, but with respect to the set $\{\,\phi > \psi\,\}$. Fixing $\omega_\phi^{n\text{-}1}$ (or $\omega_\psi^{n\text{-}1}$) and arguing the same way we obtain

$$(1/q)^{1/n}\int_{\{\,\phi>\psi\,\}} \omega_\phi^n \leq \int_{\{\,\phi>\psi\,\}} \omega_\phi^{n\text{-}1} \wedge \omega. \tag{3.13}$$

But adding these inequalities and the assumption that $A_0$ is massless one obtains

$$(1/q)^{1/n} = (1/q)^{1/n}\int_{\{\,\phi>\psi\,\}} \omega_\phi^n + (1/q)^{1/n}\int_{\{\,\phi<\psi\,\}} \omega_\phi^n \leq$$

$$\leq \int_{\{\,\phi>\psi\,\}} \omega_\phi^{n\text{-}1} \wedge \omega + \int_{\{\,\phi<\psi\,\}} \omega_\phi^{n\text{-}1} \wedge \omega \leq 1,$$

a contradiction.

So, we can assume that the whole mass of $\mu$ is concentrated on $\{ \phi = \psi \} \neq X$.

Now we prove inductively that the same holds (i.e. the whole mass is concentrated on $A_0$) for any of the measures

$$\omega_\phi^k \wedge \omega_\psi^s \wedge \omega^{n-k-s}, \; k, \; s \in \{ 0, \cdots, n-1 \}, \; 0 \leq k+s \leq n.$$

The case $k+s=n$ is already shown above.

Suppose the claim is proven for all $k$ and $s$ such that $k+s=r+1$. Below we prove it (for all cases) such that $k+s=r$.

Let $\phi_j := max\{ \phi, -j \}$. Fix $t \in (0,1)$. Consider the sets

$$V_{t,j} := \{ \phi + (t/j)\phi_j + (3/2)t < \psi \} \subset \{ \phi < \psi \}.$$

Again, by the comparison principle we obtain

$$\int_{V_{t,j}} {\omega_\phi}^k \wedge {\omega_\psi}^s \wedge (\omega_\psi + (t/j)\omega) \wedge \omega^{n-r-1} \leq \int_{V_{t,j}} {\omega_\phi}^k \wedge {\omega_\psi}^s \wedge (\omega_\phi + (t/j)\omega_{\phi_j}) \wedge \omega^{n-r-1}.$$

Now, by induction hypothesis (and set inclusions) the first terms on both sides vanish and the equation above reads

$$\int_{V_{t,j}} {\omega_\phi}^k \wedge {\omega_\psi}^s \wedge \omega^{n-r} \leq \int_{V_{t,j}} {\omega_\phi}^k \wedge {\omega_\psi}^s \wedge \omega_{\phi_j} \wedge \omega^{n-r-1}.$$

Note that $V_{t,j}$ is a decreasing sequence of sets in terms of $j$. Letting $j \to \infty$ and using vanishing on pluripolar sets we obtain

$$\int_{\{ \phi + (3/2)t < \psi \}} {\omega_\phi}^k \wedge {\omega_\psi}^s \wedge \omega^{n-r} \leq \int_{\{ \phi + (3/2)t < \psi \}} {\omega_\phi}^{k+1} \wedge {\omega_\psi}^s \wedge \omega^{n-r-1} = 0,$$

where the last equality again follows from the induction hypothesis. Finally letting $t \searrow 0$ we obtain

$$\int_{\{ \phi < \psi \}} {\omega_\phi}^k \wedge {\omega_\psi}^s \wedge \omega^{n-r} \leq \int_{\{ \phi < \psi \}} = {\omega_\phi}^{k+1} \wedge {\omega_\psi}^s \wedge \omega^{n-r-1} = 0.$$

Again exchanging $\phi$ with $\psi$ and $\{ \phi < \psi \}$ with $\{ \phi > \psi \}$ one obtains that the measures ${\omega_\phi}^k \wedge {\omega_\psi}^s \wedge \omega^{n-r}$ are massless on $\{ \phi \neq \psi \}$. This finishes the proof of the induction step.

Notice that in particular for $k = s = 0$ we obtain

$$\int_{\{ \phi \neq \psi \}} \omega^n = 0.$$

But $\{ \phi \neq \psi \}$ is a pluri-fine open set, and $\omega^n$ is equicontinuous with the Lebesgue measure. So a (non empty) pluri-fine open set would have zero Lebesgue measure, a contradiction Theorem 2.1.23. This shows that our initial assumption $\phi - \psi \neq const$ is false. □

3.2.5. *Stability of solutions.* The study of PDE's stability, roughly speaking, consists of the following: given an equation with fixed data, we perturb the relevant parameters in a controlled way and analyze the deviation of the solution of the perturbed equation from the initial one. Usually stability is the first step towards regularity theory of solutions.

The basic (but fundamental) observation is that stability breaks down if there is no uniqueness of solutions of the considered problem. Indeed, take two different solutions with the same data and consider the second one as a solution of perturbed problem with "zero" perturbation. This clearly shows that there is no way to controll deviation of solutions in terms of the deviation of the parameters.

It should be noted that it is usually the case in PDE theory that regularity is much more difficult when there is no uniqueness. However, we have already shown uniqueness of the Monge-Ampère operator on compact Kähler manifolds for a very general class of functions. It is therefore meaningfull to pose the stability question.

Below we state three stability results. The first is due to Błocki, the second was proved by Eyssidieux, Guedj and Zeriahi and the third is due to Kołodziej.

**Theorem 3.2.14** ([Bl5]). *Let* $u,\ v \in PSH(X,\omega) \cap L^{\infty}(X)$ *solve the equations*

$$(\omega + dd^c u)^n = f\omega^n,\ (\omega + dd^c v)^n = g\omega^n,$$

*where* $f$ *and* $g$ *are nonnegative functions satisfying* $\int_X f\omega^n = \int_X g\omega^n = \int_X \omega^n$. *Assume that the solutions are normalized so that*

$$\int_X u\omega^n = \int_X v\omega^n.$$

*Then there exists a constant* $C$ *depending on* $X$, $sup_X u$ *and* $sup_X v$, *such that*

$$||u - v||_{L^{\frac{2n}{n-1}}} \leq C||f - g||_{L^1},$$

*(norms are taken with respect to the density* $\omega^n$*).*

This shows that among the bounded solutions, small deviation of the function $f$ (perturbation in $L^1$) forces the perturbed solution to be close in $L^{\frac{2n}{n-1}}$ norm (provided we controll the $L^{\infty}$ norms of the solutions).

**Theorem 3.2.15** ([EGZ]). *Let* $u,\ v \in PSH(X,\omega)$ *solve the equations*

$$(\omega + dd^c u)^n = f\omega^n,\ (\omega + dd^c v)^n = g\omega^n,$$

*where* $f$ *and* $g$ *are nonnegative functions satisfying* $\int_X f\omega^n = \int_X g\omega^n = \int_X \omega^n$ *and moreover* $f,\ g \in L^p(\omega^n),\ p > 1$. *Then there exists a constant* $C$ *dependent on* $X,\ p,\ s,\ \epsilon,\ ||f||_{L^p},\ ||g||_{L^p}$ *such that*

$$||u - v||_{\infty} \leq C||u - v||_{L^s(\omega^n)}^{\frac{s}{nq+s+\epsilon}}, \forall s > 0,\ \epsilon > 0.$$

Note that this stability result requires more regularity (we need to know that the Monge-Ampère measures are in $L^p$) but also gives us significantly more: now the perturbed solution is uniformly close to the original one. It should be also noted that in [EGZ] only the case $s = 2$ was proven but the general case follows in the same lines.

**Theorem 3.2.16** ([K3]). *In the same setting as in the previous theorem there exists a constant*
$c = c(X,\ p,\ \epsilon,\ c_0)$, *where* $c_0$ *is an upper bound for* $||f||_p$ *and* $||g||_p$, *such that*

$$||u - v||_{\infty} \leq c||f - g||_1^{\frac{1}{n+3+\epsilon}},$$

*provided one normalizes* $u$ *and* $v$ *so that* $\sup_X(u - v) = \sup_X(v - u)$.

(Recall that the assumption $f,\ g \in L^p,\ p > 1$ forces boundedness of solutions $u, v$). Although the normalization here is somewhat strange, the stability result is quite important since it deals merely with the Monge-Ampère densities of $u$ and $v$, which quite often is the only information we have about the functions.

Below we sketch a proof of this important result. Basically, all the following argument is directly quoted from [K3]. Recall that by the $L^{\infty}$ estimate, there exists a constant $c(X, B)$ dependent only on $X$ and on the bound for the norm $||f||_p \leq B$, such that $sup_X\phi - \inf_X \phi \leq c(X, B)$. Stability is an immediate consequence of the following theorem:

**Theorem 3.2.17.** *Let $\phi$, $\psi$, $f$, $g$ be as above. Fix $A > 0$, such that $||f||_p \leq A$, $||g||_p \leq A$. Let also $a = c(X, \frac{3}{2}A)$ and the functions $Q : \mathbb{R}_{>0} \to \mathbb{R}$ and $\kappa : \mathbb{R}_{>0} \to \mathbb{R}$ be defined as in Lemma 3.2.3. Define also $\gamma(t) = D\kappa^{-1}(t)$ ($D$ is some non negative constant - we fix it, as in* [K3]*, to be $(\frac{2a}{a+1})^n \frac{(\frac{3}{2})^{\frac{1}{n}}-1}{3}$ however we wish to emphasize that its value is immaterial - it serves merely as a normalization condition). $\kappa^{-1}(t)$ is the inverse of the function $\kappa$.*

*Under these assumptions if $\|f - g\|_{L^1} \leq \gamma(t)t^{n+3}$ then*

$$\|\phi - \psi\|_{L^\infty} \leq Ct$$

*for $t < t_0$, where $t_0 > 0$ depends on $\gamma$, and $C$ depends on the $L^p$-norms of $f$ and $g$.*

Indeed suppose this result were true. Fix $\epsilon > 0$. Choose also $Q(t) = c_m t^m$ for $m = \frac{n^2}{\epsilon}$. Then $\gamma(t)t^{n+3} = ct^{n+3+\epsilon}$ is increasing on $[0, t_0)$. If for some $t_1 \in [0, t_0)$, we have $||f-g||_1 = \gamma(t_1)t_1^{n+3}$ then $||\phi-\psi||_\infty \leq C_1 t_1 \leq C_2||f-g||_1^{\frac{1}{n+3+\epsilon}}$. On the other hand is such $t_1$ does not exist (i.e. $||f-g||_1 \geq \gamma(t_0)t_0^{n+3}$) then $||\phi-\psi||_\infty \leq 2a(\frac{||f-g||_1}{\gamma(t_0)t_0^{n+3}})^{\frac{1}{n+3+\epsilon}} \leq c(a,\epsilon)||f-g||_1^{\frac{1}{n+3+\epsilon}}$. Thus stability holds with the constant $c = max\,\{\,C_2,\ c(a,\epsilon)\,\}$.

Below we give the proof of Theorem 3.2.17.

Assume, without loss of generality, that $\int_{\{\psi<\phi\}}(f+g)\omega^n \leq 1$, (if this was not hte case the exchange the roles of $\psi$ and $\phi$).

By adding constants to both $\phi$ and $\psi$, (this does not affect the pertinent parameters), we can also assume that $0 \leq \phi \leq a$.

Since $lim_{t\to 0}\gamma(t) = 0$, by the definition of $\kappa$ we can fix $0 < t_0 < 1$ small enough, so that $\gamma(t_0){t_0}^{n+3} < \frac{1}{3}$, (clearly this will also hold for $0 < t < t_0$).

Fix such a $t$ and define the set $E_k = \{\psi < \phi - kat\}$.

The following estimate holds

$$\int_{E_0} g\omega^n = \frac{1}{2}\int_{E_0}\big((f+g)+(g-f)\big)\omega^n \leq \frac{1}{2}(1+\frac{1}{3}) = \frac{2}{3}.$$

Consider the function $g_1$, defined as $\frac{3g}{2}$ on $E_0$ and some non negative constant on the complement. By the above estimate, one can choose this constant so that $\int_X g_1\omega^n = 1$ and the $L^p$-norm is bounded by $\frac{3A}{2}$.

Thus there exists a continuous solution $\rho \in PSH_\omega(X)$ to the problem

$${\omega_\rho}^n = g_1\omega^n,\ max_X\rho = 0,$$

where $||\rho||_\infty \leq a$, since $||g_1||_p \leq \frac{3}{2}A$.

Observe that $-2at \leq -t\phi + t\rho \leq 0$, thus we obtain the chain of inclusions

$$E_2 \subset E := \{\psi < (1-t)\phi + t\rho\} \subset E_0.$$

Let $G$ be the set $\{f < (1-t^2)g\}$. Then on $E_0 \setminus G$, we get

$$\big((1-t^2)^{-\frac{1}{n}}\omega_\phi\big)^n \geq g\omega^n,\ \big((\frac{3}{2})^{-\frac{1}{n}}\omega_\rho\big)^n = g\omega^n.$$

By the inequality for mixed Monge-Ampère measures (the version from [K3] is sufficient here) it follows that on $E_0 \setminus G$,

$$(\frac{3}{2})^{-\frac{n-k}{n}}(1-t^2)^{-\frac{k}{n}}{\omega_\phi}^k \wedge {\omega_\rho}^{n-k} \geq g\omega^n.$$

Let $q = (\frac{3}{2})^{\frac{1}{n}} > 1$. The above estimate can be written as:

$$\omega_\phi^k \wedge {\omega_\rho}^{n-k} \geq q^{n-k}(1-t^2)^{\frac{k}{n}}g\omega^n.$$

Thus

$$\begin{aligned}
{\omega_{t\rho+(1-t)\phi}}^n &\geq \big((1-t)(1-t^2)^{\frac{1}{n}} + qt\big)^n g\omega^n \\
&\geq \big((1-t)(1-t^2) + qt\big)^n g\omega^n \\
&\geq \big(1 + t(q-1) - t^2\big) g\omega^n \\
&\geq \big(1 + \frac{t}{2}(q-1)\big) g\omega^n.
\end{aligned} \tag{3.14}$$

By the definition of $G$ and the assumptions we obtain

$$t^2 \int_G g\omega^n \leq \int_G (g-f)\omega^n \leq \gamma(t)t^{n+3},$$

which reads:

$$\int_G g\omega^n \leq \gamma(t)t^{n+1}. \tag{3.15}$$

Thus we get

$$\begin{aligned}
\big(1 + \frac{t}{2}(q-1)\big) \int_{E\setminus G} g\omega^n &\leq \int_E {\omega_{t\rho+(1-t)\phi}}^n \\
&\leq \int_E {\omega_\psi}^n \\
&\leq \int_{E\setminus G} g\omega^n + \gamma(t)t^{n+1}.
\end{aligned} \tag{3.16}$$

(the second inequality above is justified by the comparison principle).

Thus

$$\frac{q-1}{2} \int_{E\setminus G} g\omega^n \leq \gamma(t)t^n.$$

The set inclusion $E_2 \subset E$ implies that

$$\frac{q-1}{2}\Big(\int_{E_2} g\omega^n - \gamma(t)t^{n+1}\Big) \leq \frac{q-1}{2}\Big(\int_{E_2} g\omega^n - \int_G g\omega^n\Big) \leq \frac{q-1}{2} \int_{E\setminus G} g\omega^n \leqslant \gamma(t)t^n,$$

leading to the inequality

$$\int_{E_2} g\omega^n \leq \Big(t + \frac{2}{q-1}\Big)\gamma(t)t^n \leq \frac{3}{q-1}\gamma(t)t^n$$

for small $t$.

On the other hand, by Theorem 3.1.6 we have that

$$Cap_\omega(E_4) \leq \Big(\frac{a+1}{2at}\Big)^n \int_{E_2} g\omega^n.$$

Coupling thes results we obtain

$$Cap_\omega(E_4) \leq \Big(\frac{a+1}{2a}\Big)^n \frac{3}{q-1}\gamma(t).$$

The, if the set $E' := \{\psi < \phi - (4a+2)t\}$ was nonepmty, we would get:

$$2t \leq \kappa(Cap_\omega(E_4)) \leq \kappa\Big(\Big(\frac{a+1}{2a}\Big)^n \frac{3}{q-1}\gamma(t)\Big) = t,$$

a contradiction for $t > 0$. Thus $\psi \geqslant \phi - (4a+2)t$, which confirms our claim.

Since stability estimates can be used in the further study of regularity of solutions it is important to obtain *sharp* estimates. In particular it is important to know whether exponents in the results above are sharp. As example 3.2.19 will show, the stability exponents in Eyssidieux, Guedj and Zeriahi theorem are sharp in some cases. The Kołodziej's stability exponent, however, can be improved to (optimal) $\frac{1}{n+\epsilon}$ and the proof of this result, taken from [DZ], will be given below.

*Proof.* Before we proceed further we make a small improvement of the stability exponent in Kolodziej's theorem which is relatively easy:

Note that in the definition of set $G = \{f < (1-t^2)g\}$ one can exchange $t^2$ with $\frac{t}{b}$ ($b$ is a large constant independent of the involved parameters), and the same argument still goes through (except that in the forelast step instead of $E_4$ we take the set $E_{2+s}$ for $s$ big enough depending only on $b$, such that using the Proposition 3.1.7 we can kill the contant in front of $\gamma(t)t^n$). Thus from $||f-g||_1 \leq \gamma(t)t^{n+2}$ we obtain $||\phi - \psi||_\infty \leq Ct$. In particular the stability result holds with exponent $\frac{1}{n+2+\epsilon}$.

Further improvements, however, are nontrivial, and therefore we first describe the strategy of proof and then technical details will be provided.

Trying to improve the exponent, one has to follow the main steps of the original proof and carefully analyze points where an exponent loss occurs. Therefore we shall iterate the original argument, defining at each step new function $\rho$ and use the previous step to get estimates for $||\rho - \psi||_\infty$, which in turn will be used to choose the new set $E$ in a "better" way. Each time the exponent will improve a bit (while keeping all the relevant quantities under control). In the limit we would get the desired optimal exponent.

The first step of the iteration is Kołodziej's argument (with the improvement yielding exponent $\frac{1}{n+2+\epsilon}$) and in the sequel this will will be often denoted as Step 1.

Now the iteration procedure form Step $k$ to Step $k+1$ goes as follows:

The mechanism is based on the fact that $||f-g||_1 \leq \gamma(t)t^{\beta_k}$ (so, in the improved original proof in Step 1 we have $\beta_1 = n+2$) yields $\int_{\{\psi+mt<\phi\}}(\omega_\psi)^n \leq c_0 t^n$ for some constant $m = m(k)$ and $c_0$ (in what follows $c_i$ denote constants independent of the relevant quantities). So we try to find $\beta$ as small as possible for which this implication holds true with uniform control on $c_0$ and enlarging $m$ if needed.

So assume that the $k$-th Step has been terminated and consider the inequality

$$||f-g||_1 \leq \gamma(t)t^{\beta_{k+1}},\ t<1.$$

Then if $l := t^{\frac{\beta_{k+1}}{\beta_k}}$, $\beta_{k+1} < \beta_k$, we obtain $||f-g||_1 \leq \gamma(l)l^{\beta_k}$, so from Step $k$ we know that

$$\int_{E_r} g\omega^n \leq \gamma(l)l^n, \tag{3.17}$$

where, as before, $E_r := \{\psi < \phi - rat\}$, and $r = r(k)$ is a constant under control, whose existence is granted by the termination of the $k$-th Step. Hence

$$\int_{E_r} g\omega^n \leq c_1 t^{\frac{\beta_{k+1}n}{\beta_k}},\ t \leq t_0 \tag{3.18}$$

(recall $\gamma(t)$ decreases to 0, as $t \searrow 0$).

Now fix a small positive constant $\delta_k$ to be choosen later on.

Consider the "new" function

$$g_1(z) = \begin{cases} (1+\frac{t^{\delta_k}}{2})g(z), & z \in E_{r(k)} \\ c_2 g(z), & z \in X \setminus E_{r(k)}, \end{cases}$$

where $0 \le c_2 \le 1$ is choosen such that $\int_X g_1\omega^n = 1$. (The constant $\frac{1}{2}$ in front of $t^{\delta_k}$ is taken to assure that the integral over $E_{r(k)}$ is less than 1. Note that despite the fact that the case $t$ being small is of main interest, when $\delta_k$ is also small the quantity $t^{\delta_k}$ cannot be controlled by a constant smaller than 1). As in Step 1 we find a solution $\rho$ (dependent on $k$) to the problem $(\omega_\rho)^n = g_1\omega^n$, $max_X\rho = 0$. Again $\rho \ge -a$ and we renormalize $\rho$ by adding a constant so that $max_X(\psi - \rho) = max_X(\rho - \psi)$ (this can by done in an uniform way).

Now by Step $k$

$$||\rho - \psi||_\infty \le c_3||g - g_1||_1^{\frac{1}{\beta_k+\epsilon}} = c_3([\int_{E_{r(k)}} + \int_{X\setminus E_{r(k)}}]|g - g_1|\omega^n)^{\frac{1}{\beta_k+\epsilon}} =$$

$$= c_3(2t^{\delta_k}\int_{E_{r(k)}} g\Omega)^{\frac{1}{\beta_k+\epsilon}} \le c_4 t^{\frac{\delta_k + \frac{\beta_{k+1}n}{\beta_k}}{\beta_k+\epsilon}}.$$

If $\delta_k$ is sufficiently small the last exponent is less than 1 and we define $\alpha_k := 1 - \frac{\delta_k + \frac{\beta_{k+1}n}{\beta_k}}{\beta_k+\epsilon}$.

Then by the above estimate

$$E_s = \{\, \psi + sat < \phi \,\} = \{(1 - \frac{1}{2}t^{\alpha_k})(\psi + sat) < (1 - \frac{1}{2}t^{\alpha_k})\phi \,\} \subset \tag{3.19}$$

$$\subset \{\, \psi < (1 - \frac{1}{2}t^{\alpha_k})\phi + \frac{1}{2}t^{\alpha_k}\rho + \frac{1}{2}c_4 t - sat(1 - \frac{1}{2}t^{\alpha_k}) \,\} =: E \subset$$

$$\subset \{\, \psi < (1 - \frac{1}{2}t^{\alpha_k})\phi + \frac{1}{2}t^{\alpha_k}\psi + c_4 t - sat(1 - \frac{1}{2}t^{\alpha_k}) \,\} =$$

$$= \{\, \psi + (s - \frac{c_4}{1 - \frac{1}{2}t^{\alpha_k}})at < \phi \,\} \subset E_m,$$

provided $s \ge \frac{2c_4}{a} + m$, (the constant $\frac{1}{2}$ is again added in order to control $1 - \frac{1}{2}t^{\delta_k}$ from below).

Consider the "new" set

$$G_1 := \{\, f < (1 - \frac{t^{\alpha_k + 3\delta_k}}{8n2^{\frac{n-1}{n}}})g \,\}.$$

Using that $h(t) = (1 + \frac{t^{\delta_k}}{2})^{\frac{1}{n}} - 1 - \frac{1}{4n2^{\frac{n-1}{n}}}t^{2\delta_k}$ is increasing in $[0,1]$ and hence nonnegative there, we conclude as in Step 1, że na $E_m \setminus G$

$$(\omega_{\frac{1}{2}t^{\alpha_k}\rho + (1-\frac{1}{2}t^{\alpha_k})\phi})^n \ge ((1 - \frac{1}{2}t^{\alpha_k})(1 - \frac{t^{\alpha_k+3\delta_k}}{8n2^{\frac{n-1}{n}}})^{\frac{1}{n}} + (1 + \frac{t^{\delta_k}}{2})^{\frac{1}{n}}\frac{1}{2}t^{\alpha_k})^n g\omega^n \ge \tag{3.20}$$

$$\ge ((1 - \frac{1}{2}t^{\alpha_k})(1 - \frac{t^{\alpha_k+3\delta_k}}{8n2^{\frac{n-1}{n}}}) + (1 + \frac{1}{4n2^{\frac{n-1}{n}}}t^{2\delta_k})\frac{1}{2}t^{\alpha_k})^n g\omega^n \ge (1 + \frac{1}{2}\frac{t^{\alpha+2\delta_k}}{8n2^{\frac{n-1}{n}}})g\omega^n.$$

As in Step 1 on $G$ we obtain

$$\frac{t^{\alpha_k+3\delta_k}}{8n2^{\frac{n-1}{n}}}\int_G g\omega^n \le \int_G (g - f)\omega^n \le \gamma(t)t^{\beta_{k+1}}, \tag{3.21}$$

so, using (3.20), (3.21) and the comparison principle we obtain

$$(1 + \frac{t^{\alpha_k+2\delta_k}}{16n2^{\frac{n-1}{n}}})\int_{E\setminus G} g\omega^n \le \int_E (\omega_{(1-t^{\alpha_k})\phi + t^{\alpha_k}\rho})^n \le \tag{3.22}$$

$$\le \int_E g\omega^n \le \int_{E\setminus G} g\omega^n + c_5\gamma(t)t^{\beta_{k+1}-\alpha_k-3\delta_k}.$$

Finally, as in Step 1, we get

$$\int_{E_s\setminus G} g\omega^n \leq \int_{E\setminus G} g\omega^n \leq c_6\gamma(t)t^{\beta_{k+1}-2\alpha_k-5\delta_k}$$

and

$$\int_{E_s} g\omega^n \leq c_7\gamma(t)t^{\beta_{k+1}-2\alpha_k-5\delta_k}.$$

If $\beta_{k+1}-2\alpha_k-5\delta_k = n$, we can proceed as in Step 1 to get $\max(\phi-\psi) = \max(\psi-\phi) \leq (2s+2)t$, and $||\phi-\psi||_\infty \leq C(\epsilon)||f-g||_1^{\frac{1}{\beta_{k+1}+\epsilon}}$, $\forall\epsilon > 0$. Of course, we have to adjust here the function $\kappa$ to the constant $c_7$. However even the (hypothetical) overgrowth of $c_7$ may be handled for small $t$ at the expense on an epsilon loss of the exponent. Since this does not affect the argument significantly, we omit this step in the sequel for the sake of brewity.

Now $\beta_{k+1}-2\alpha_k-5\delta_k = n$ yields

$$\beta_{k+1}(1+\frac{2n}{\beta_k(\beta_k+\epsilon)}) = n+2+5\delta_k-2\frac{\delta_k}{\beta_k+\epsilon}. \tag{3.23}$$

Note that $\delta_k$ can be made arbitrarily small. Also if $\beta_k > n+\epsilon_0$ for some $\epsilon_0 >> \epsilon$), we get that $\beta_{k+1} < \beta_k$ thus the stability exponent increases.

Thus, in the Step $k+1$ we have defined new constants $\delta_k,\ m_{k+1},\ \beta_{k+1}$, such that:

(1) $\delta_k$ is sufficiently small so that, from (3.23) one can get $n < \beta_{k+1} < \beta_k$ and $\alpha_k < 1$,
(2) the constant $m_{k+1}$ (i.e. $s$ in the reasoning above) is under control and moreover $\int_{\{\psi+m_{k+1}at<\phi\}}(\omega+dd^c\psi)^n \leq c_0t^n$.

This finishes Step $k+1$.

Thus $\beta_k$ is a convergent sequence. Assume also that $\delta_k \searrow 0$. THen if $A$ is the limit of the sequence $\{\beta_k\}$, we obtain

$$A(1+\frac{2n}{A(A+\epsilon)}) = n+2 \Rightarrow A = \frac{n+2-\epsilon+\sqrt{(n-2+\epsilon)^2+8\epsilon}}{2}.$$

If $\epsilon \to 0^+ \Rightarrow A \to n$, so $\beta_k$ are arbitrarily close to $n$, provided $k$ is big enough and $\epsilon$ is chosen small independently of the recursion steps. □

**Remark 3.2.18.** *The proof above can be repeated in the case of big forms without any changes. Therefore stability also holds in the case of Monge-Ampère equations in the big case.*

The following example, taken from [DZ], shows that the obtained exponent is sharp:

**Example 3.2.19.** *Fix appropriate positive constants $B,\ D$, such that $D < B$ and $B2^{2\alpha} < \log 2 + D$ for some fixed $\alpha \in (0,1)$ (such constants clearly exist). Then the function*

$$\widehat{\rho}(z) := \begin{cases} B||z||^{2\alpha}, & ||z|| \leq 1 \\ \max\{B||z||^{2\alpha}, \log(||z||)+D\}, & 1 \leq ||z|| \leq 2 \\ \log(||z||)+D, & ||z|| \geq 2 \end{cases}$$

*is well defined, plurisubharmonic in $\mathbb{C}^n$ and of logarithmic growth. One can smooth out $\widehat{\rho}$, so that the new function $\rho$ is again of logarithmic growth, radial, smooth away from the origin and $\rho(z) = B||z||^{2\alpha}$ for $||z|| \leq \frac{3}{4}$.*

*This smoothing can be obtained in many ways. It is enough to observe that the function*

$$m(x,y) = max(x,y),$$

*can be approximated by a decreasing sequence of smooth convex functions $m_\epsilon$, such that $m_\epsilon(x,y) = x$ if $x - y \geq \epsilon$ and $m_\epsilon(x,y) = y$ if $y - x \geq \epsilon$. Thus if $\epsilon$ is small enough, then exchanging the maximum int the definition of $\widehat{\rho}$ by $m_\epsilon$ we get the claimed smoothing. More details concerning this technique can be found in* [Si2] *and* [De3].

*Via the standard inclusion*

$$\mathbb{C}^n \ni z \longrightarrow [1:z] \in \mathbb{P}^n$$

*one identifies $\rho(z)$ with*

$$\overline{\rho}([z_0 : z_1 :, \cdots, : z_n]) := \rho(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}) - \frac{1}{2}\log(1 + \frac{||z||^2}{|z_0|^2}) \in PSH(\mathbb{P}^n, \omega_{FS})$$

*(here $\omega_{FS}$ is the Fubini-Study metric on $\mathbb{P}^n$, and the values of $\overline{\rho}$ on the hypersurface $\{z_0 = 0\}$ are understood as limits of values of $\overline{\rho}$ when $z_0$ approaches $0$.) It is clear that $\omega_{\overline{\rho}}^n = (dd^c\rho)^n$ in the chart $z_0 \neq 0$ and in fact one can neglect what happens on the hypersurface at infinity.*

*Now for a vector $h \in \mathbb{C}^n$ one can define $\rho_h(z) := \rho(z+h)$ and analogously the corresponding $\overline{\rho}_h$. Note that when $||h|| \to 0$, we get $\overline{\rho}_h \rightrightarrows \overline{\rho}$.*

*Observe that*

$$B||h||^{2\alpha} \leq ||\overline{\rho}_h - \overline{\rho}||_\infty. \tag{3.24}$$

*The Monge-Ampère measures of $\overline{\rho}$ and $\overline{\rho}_h$ are smooth functions except at the origin and $-h$, respectively and belong to $L^p(\omega_{FS}^n)$, for some $p > 1$ dependent on $\alpha$.*

*Now $\int_{\mathbb{P}^n} |\omega_{\overline{\rho}}^n - \omega_{\overline{\rho}_h}^n| = \int_{\mathbb{C}^n} |(dd^c\rho)^n - (dd^c\rho_h)^n|$. To estimate the last term we divide $\mathbb{C}^n$ into three pieces (we suppose $||h||$ is small):*

$$\int_{\mathbb{C}^n} |(dd^c\rho)^n - (dd^c\rho_h)^n| = \int_{\{||z|| \leq 2||h||\}} + \int_{\{2||h||<||z||\leq \frac{1}{2}\}} + \int_{\{||z||>\frac{1}{2}\}}.$$

*Using the fact that $\rho$ and $\rho_h$ are smooth functions in a neighbourhood of $\{||z|| > \frac{1}{2}\}$ one can easily estimate the last term by $||h||C_0$ for some constant independent of $h$. For the first two terms we observe that $(dd^c\rho)^n = B^n||z||^{2n(\alpha-1)}$, $(dd^c\rho_h)^n = B^n||z+h||^{2n(\alpha-1)}$.*

*Now we use a computation trick which can be found in* [KW].

$$\begin{aligned}
&\int_{\{||z||\leq 2||h||\}} |(dd^c\rho)^n - (dd^c\rho_h)^n| = \\
&= B^n \int_{\{||z||\leq 2||h||\}} |||z||^{2n(\alpha-1)} - ||z+h||^{2n(\alpha-1)}| \leq \\
&\leq 2B^n \int_{\{||z||\leq 3||h||\}} ||z||^{2n(\alpha-1)} = C_1||h||^{2n\alpha}.
\end{aligned}$$

*We estimate the second term as follows:*

$$\begin{aligned}
&\int_{\{2||h||\leq||z||\leq\frac{1}{2}\}} |(dd^c\rho)^n - (dd^c\rho_h)^n| = \\
&= B^n \int_{\{2||h||\leq||z||\leq\frac{1}{2}\}} |||z||^{2n(\alpha-1)} - ||z+h||^{2n(\alpha-1)}| \leq \\
&\leq B^n \int_{2||h||<||z||} |\int_0^1 < \nabla||z+th||^{2n(\alpha-1)}, h > |dt \leq \\
&\leq C_2||h|| \int_{||h||<||z||} ||z||^{2n(\alpha-1)-1} \leq C_3||h||^{2n\alpha},
\end{aligned}$$

*provided $\alpha < \frac{1}{2n}$, so that the integral is finite. Finally we obtain for small $||h||$*

$$\int_{\mathbb{P}^n} |\omega_{\rho}^n - \omega_{\rho_h}^n| \leq C_1||h||^{2n\alpha} + C_3||h|| \leq C_4||h||^{2n\alpha}. \tag{3.25}$$

*Suppose now that we have a stability estimate $||\phi - \psi||_\infty \leq C_5||f-g||_1^{\frac{1}{m}}$. Then coupling (3.24) and (3.25) one gets*

$$||h||^{2\alpha} \leq C_6(||h||^{2n\alpha})^{\frac{1}{m}}, \ \alpha \in (0, \frac{1}{2n}).$$

*If we let $||h|| \to 0$ this can hold only if $m \geq n$.*

*Finally, using the same example and similar estimates one can show that the Eyssidieux, Guedj and Zeriahi exponent is also sharp, provided that $p < 2$ and $s > \frac{2np}{2-p}$ (the reason for these obstructions is that the second integral we estimate as in the example would be divergent otherwise). It is, however, very likely that these exponents are sharp in general.*

3.2.6. *Regularity of solutions.* As we have already mentioned, stability is usually useful in proving higher regularity of solutions of specific PDE's. Thus the results in the previous subsections suggest that we can expect better regularity properties for the solutions. A recent result, due to Kołodziej confirms these expectations:

**Theorem 3.2.20** ([K5]). *Let $\phi \in PSH(X,\omega)$ solve the Dirichlet problem*

$$(\omega + dd^c \phi)^n = f\omega^n, \ sup_X\phi = 0, \ \phi \in PSH(X,\omega),$$

*where $f \geq 0$ is a function from the space $L^p(\omega^n)$, $p > 1$. Then $\phi$ is Hölder continuous, with Hölder exponent dependent on p, the dimension of $X$ and on its geometry.*

Below we sketch the main ideas of the proof of this reult.

Recall that locally (in a coordinate chart) $\phi$ can be approximated via convolutions. Using the continuity of $\phi$ (which was proven in earlier subsections) and the (complicated) Richberg technique one can glue these local approximants, so that one can obtain a sequence of global $\omega$-psh functions converging to $\phi$. This process of patching local data is affected by the local geometry of the manifold and this is the place where dependence on $X$ appears. The rate of convergence in local $L^1$ norms, in turn, depends on the laplacian of $\phi$ (due to Jensen formula from potential theory). By these results one can control the rate of convergence in $L^\infty$ norm, away from sets of small Lebesgue measure.

On the other hand, due to Theorem 3.2.16 (or its generalization) the solution of the Dirichlet problem

$$\psi \in PSH(X,\omega), \ (\omega + dd^c \psi)^n = g\omega^n, \ sup_X(\phi - \psi) = sup_X(\psi - \phi),$$

where

$$g(z) = \begin{cases} 0, \ for \ z \in E, \\ cf \ for \ z \in X \setminus E \end{cases}$$

(the constant $c$ is chosen so that $\int_X g\omega^n = \int_X \omega^n$) is, for small sets $E$, uniformly close to $\phi$.

Coupling these two ways of approximation of $\phi$ one can prove that too slow convergence in $L^\infty$ norm of the patched-from-local-data sequence would lead to contradiction with the vanishing of $g$ on (suitably chosen) small set $E$. Details of the proof can be found in [K5].

In a particular case when $X$ is *homogeneous* i.e. when the group of automorphisms preserving $\omega$ acts transitively on $X$ (for example this is the case for the manifold $\mathbb{P}^n$ equipped with the Fubini-Study form) Eyssidieux, Guedj and Zeriahi in [EGZ] have proved that the Hölder exponent is independent of the geometry $X$, and it depends merely on $n = dimX$ and $p$ (one can take any exponent smaller than $\frac{2}{2+\frac{np}{p-1}}$). Their proof is based on approximation of $\phi$ by $\phi_h$ - a composition of $\phi$ with an automorphism close to the identity (an analogue of a shift by a small vector in $\mathbb{C}^n$). Then Theorem 3.2.15 is used and finally an easy estimate of the $L^2$ norm of the difference $\phi_h - \phi$ completes the proof. We refer to [EGZ] for the details.

These results lead to natural question of precise dependence of the optimal Hölder exponent in terms of the geometry of $X$ (if such a dependence indeed exists). Perhaps even more interesting is the question of generalizing Kołodziej's theorem in the case of big forms (of course this is meaningful only if the continuity statement is true). As we shall see in the next section a positive solution of these problems would lead to very interesting applications in geometry.

## 4. Applications in geometry

4.1. **The Kähler-Ricci flow - behavior at critical times.** The Kähler-Ricci flow is a parabolic flow, which is a complex counterpart of the more familiar Ricci flow. The latter has become one of the main objects of study in Riemannian geometry and, by works of Hamilton, Perelman and other authors, has found spectacular applications such as the solution of the Poincare conjecture.

**Definition 4.1.1 ((Normalized) Kähler-Ricci flow).** *Let $X$ be a Kähler manifold equipped with a Kähler form $\omega$. The Kähler-Ricci flow (starting from $\omega$) is generated by the equation*

$$\frac{\partial \omega_t}{\partial t} = -Ric_{\omega_t} + \mu\omega_t,\ \omega_0 = \omega, \tag{4.1}$$

*where $\omega_t$ is a time dependent Kähler form, while $Ric_{\omega_t}$ is its Ricci form. The constant $\mu$ depends on $(X, \omega)$ and its precise meaning will be explained below.*

Let us make some heuristic observations regarding this flow. By a general PDE theory we get a short time existence, i.e. existence for $t$ close to 0. Suppose however that the flow can actually be continued up to infinity and moreover converges to some limit form $\omega_\infty$. The term $\frac{\partial \omega_t}{\partial t}$ should, at least on a discrete sequence of times, tend to a zero form. Thus we would obtain in the limit the equation

$$Ric_{\omega_\infty} = \mu\omega_\infty. \tag{4.2}$$

Note that the left hand side represents in cohomology the class $c_1(X)$, and $\omega_\infty$ should be non-negative (for starting with a kähler form $\omega$, $\omega_t$ will remain Kähler for all the time).

So, by the above heuristics we find that the constant $\mu$ should be a suitably chosen normalization constant, so that the equation (4.2) can hold, at least on cohomology level.

The following cases are geometrically interesting:

- If the anticanonical bundle $-K_X$ is ample then one can show (by a deep geometrical Kodaira embedding theorem) that the corresponding to $-K_X$ Chern class $c_1(X)$ is in fact Kähler (carries a Kähler form). Thus $c_1(X) > 0$, so in this case $\mu > 0$. Traditionally geometers study the flow with the constant $\mu = 1$ and the general case follows by a suitable rescaling argument.
- If in turn $K_X$ is ample, then $c_1(X) < 0$ and thus $\mu < 0$. Traditionally one studies the flow with the constant $\mu = -1$.
- If $K_X = 0$ (so that the first Chern class is zero) the constant $\mu$ must be zero.

- If $K_X$ is nef and big then (in some sense) $c_1(X) \leq 0$. Again a good choice is $\mu = -1$, however the equation (4.2) can hold only in a suitably defined weak sense.

**Remark 4.1.2.** *Note that a smooth solution (if such solution exists) of the equation*

$$Ric_\omega = \lambda\omega$$

*with suitably chosen constant $\lambda$ is a very special Kähler form. Such forms (or rather metrics generated by such forms) are called Kähler-Einstein metrics. These play fundamental role in complex geometry, similarly to the Einstein metrics in Riemannian geometry.*

*Of course a necessary assumtion for existence of such metrics is that $c_1(X)$ must be definite. In the cases $c_1(X) < 0$ and $c_1(X) = 0$ such a metric always exists. So, in these cases one can hope that our heuristics are indeed true. The existence in the case $c_1(X) > 0$ depends on additional assumptions, however the full classification of all those manifolds admitting Kähler-Einstein metrics with $c_1(X) > 0$ remains one of the most important unsolved problems in complex geometry. We refer to* [T], *where one can find a thorough discussion of Kähler-Einstein metrics and the problem of their existence.*

The analysis of the Kähler-Ricci flow is in some sense easier then the ordinary Ricci flow, since (4.1) man be restated as a (scalar) parabolic equation for the potentials. In order to show this, note that from (4.1) one obtains the following cohomology equation

$$[\frac{\partial \omega_t}{\partial t}] = -c_1(X) + \mu[\omega_t].$$

Let us divide the argument in two cases.

Case 1. Let $\mu \neq 0$. For simplificity we choose the representative of $c_1(X)$ to be $Ric_\omega$ - in fact one can take this representative arbitrarily. Then, from the $\partial\bar{\partial}$-lemma (we refer to a similar reasoning in the Calabi-Yau subsection) we get

$$\omega_t = \frac{Ric_\omega}{\mu} + e^{\mu t}(\omega - \frac{Ric_\omega}{\mu}) + i\partial\bar{\partial}\phi_t$$

for some potential $\phi_t$ dependent on $t$. Thus we obtain

$$\mu\omega_t - Ric_{\omega_t} = \frac{\partial \omega_t}{\partial t} = \mu e^{\mu t}(\omega - \frac{Ric_\omega}{\mu}) + \mu\frac{Ric_\omega}{\mu} - Ric_\omega + i\partial\bar{\partial}\frac{\partial \phi_t}{\partial t}$$

$$= \mu\omega_t - \mu i\partial\bar{\partial}\phi_t + i\partial\bar{\partial}\frac{\partial \phi_t}{\partial t}.$$

This yields

$$i\partial\bar{\partial}log(\frac{\omega_t^n}{\omega^n}) = i\partial\bar{\partial}\frac{\partial \phi_t}{\partial t} - \mu i\partial\bar{\partial}\phi_t.$$

Now, since $X$ is compact, any pluriharmonic function must be constant. Thus we get an equation

$$\frac{\partial \phi_t}{\partial t} = log(\frac{\omega_t^n}{\omega^n}) + \mu\phi_t + c_t,$$

where $c_t$ is some constant dependent only on $t$. By suitable normalization of the potential $\phi_t$ we can assume that $c_t = 0$. Thus we obtain the following parabolic equation

$$(4.3) \qquad \begin{cases} \omega_t^n = e^{\frac{\partial \phi_t}{\partial t} - \mu\phi_t}\omega^n \\ \phi_0 = 0. \end{cases}$$

Case 2. Niech $\mu = 0$. From the heuristics above it follows that the suitable geometric situation is $c_1(X) = 0$. Thus $[\frac{\partial \omega_t}{\partial t}] = 0$, so the cohomology class $\omega_t$ is independent of $t$.

If $\omega_t = \omega + i\partial\bar{\partial}\phi_t$ (again we use the $\partial\bar{\partial}$-lemma), and $Ric_\omega = i\partial\bar{\partial}h_\omega$ (such a potential $h_\omega$ exists, for $c_1(X) = 0$) then

$$-Ric_{\omega_t} + Ric_\omega - i\partial\bar{\partial}h_\omega = -Ric_{\omega_t} = i\partial\bar{\partial}\frac{\partial\phi_t}{\partial t}.$$

Like the first case the equation can be reformulated (with a suitable normalization of $\phi_t$) as the problem

$$\begin{cases} \omega_t^n = e^{\frac{\partial\phi_t}{\partial t} + h_\omega}\omega^n \\ \phi_0 = 0. \end{cases} \tag{4.4}$$

Thus the Kähler-Ricci flow is equivalent to a parabolic counterpart of the Monge-Ampère equation on $X$ (if $\mu \neq 0$ the underlying form also varies). As we have already mentioned in the cases $c_1(X) < 0$ and $c_1(X) = 0$ one can hope for a long time existence and convergence in the limit to a Kähler-Einstein metric. This indeed holds, as Cao has shown.

**Theorem 4.1.3** ([Cao])**.** *Let $X$ be a compact Kähler manifold with an ample or trivial canonical bundle (in terms of Chern classes $c_1(X) < 0$ or $c_1(X) = 0$, respectively). The the Kähler-Ricci flow exists for $t \in [0, +\infty)$ and converges in the limit to a smooth form $\omega_\infty$, such that $Ric_{\omega_\infty} = \mu\omega_\infty$.*

This theorem may be regarded as an alternative proof of existence of Kähler-Einstein metrics in these cases. Cao's proof is (highly) nontrivial, and his methods are based on PDE theory without using any pluripotential notions.

The case $c_1(X) > 0$ is however more interesting. Since the Kähler-Einstein metric does not always exist we cannot expect analogous behaviour of the flow. Instead one can ask whether by using the flow one can characterize those manifolds, which satisfy $c_1(X) > 0$ and admit Kähler-Einstein metrics. This question is an object of very intensive studies in the recent years.

Important results in this direction have been obtained by Chen and Tian ([CT1] and [CT2]). In these articles the authors have shown that under some geometric assumptions on the initial form $\omega$ (guaranteeing existence of Kähler-Einstein metric on $X$) the Kähler-Ricci flow indeed exists for all the time and converges to a Kähler-Einstein metric. The proof of Chen and Tian's theorem again relies on PDE theory and on some deep geometric facts.

A more recent result is an unpublished theorem of Perelman and its generalization by Tian and Zhu [TZh].

**Theorem 4.1.4** (**Perelman - Tian - Zhu theorem**)**.** *Let $X$ be a Kähler manifold, such that $c_1(X) > 0$. If $X$ admits Kähler-Einstein metric, then the Kähler-Ricci flow starting from arbitrary Kähler form representing $c_1(X)$ has long time existence and converges in a suitable sense to a Kähler-Einstein metric.*

Intuitively this solves the first part of the problem of characterization of Kähler-Einstein manifolds. The second part, i.e. the explanation how the flow behaves under absence of such metrics is still not well understood. It is worth mentioning that an important ingredient in Tian and Zhu's proof is the $L^\infty$-estimate of Kołodziej (see the Calabi-Yau subsection).

If the canonical bundle is big ($\mu = -1$) Tian and Zhang have shown in [TZ], relying on a work of Tsuji [Ts], that the Kähler-Ricci flow exists for times in the interval $[0, T)$, where $T := sup\{\, t|\ (e^{-t} - 1)c_1(X) + e^{-t}[\omega] > 0\,\}$. This means, that the flow exists in the maximal possible time (for $T' > T$ the form $Ric_{\omega_{T'}}$ does not make sense on the set

$\omega_{T'}^n < 0$). When $K_X$ is additionally nef (thus, in a weak sense, $c_1(X) \leq 0$) we obtain $T = +\infty$,and the flow exists for any $t \in \mathbb{R}_+$.

In the both cases it is important to understand the "object" $\omega_T$ which is the (suitably understood) limit of $\omega_t$ as $t \to T^-$. Of course it cannot be a smooth form for otherwise in the nef case it would be a Kähler-Einstein metric (contradicting the non-definiteness of $c_1(X)$) and in the finite $T$ case the flow could be extended for larger times than $T$ (due to parabolic PDE theory), which is impossible.

By mixing geometric and PDE techniques Tian and Zhang [TZ] have shown, that the current $\omega_T$ is a smooth form outside some analytic set along which the class $(e^{-T} - 1)c_1(X) + e^{-T}[\omega]$ degenerates. Similar results can be found in [EGZ], [CN], [ST1], [ST2], [Ts] and [To]. PDE methods, however, do not give any information what happens on this analytic set.

On the other hand for time $T$ the equation reads

$$\omega_T^n = e^{\frac{\partial u_T}{\partial t} + u_T}\omega^n.$$

If by $\widehat{\omega}_T$ we denote the form $(e^{-T} - 1)Ric_\omega + e^{-T}\omega$, then the equation has the form

$$(\widehat{\omega}_T + i\partial\bar{\partial}u_T)^n = e^{\frac{\partial u_T}{\partial t} + u_T} f(\widehat{\omega}_T)^n,$$

where the function $f$ (with a slight abusement of notation) equals $\frac{\omega^n}{(\widehat{\omega}_T)^n}$. If the form $\widehat{\omega}_T$ is semi-definite (which is often the case - see [TZ]), we obtain a typical Monge-Ampère equation with big background form and $u_T \in PSH(X, \widehat{\omega}_T)$. Since one can bound the $L^p$ norm of the right hand side in a relatively simple way for some $p > 1$ (see [TZ]), by the theory developped in Section 3 we obtain the following corollary:

**Corollary 4.1.5** ([TZ], [EGZ])**.** *The potential $u_T$ is globally bounded.*

Thus pluripotential theory allows one to understand better the behaviour of $\omega_T$ on $X$.

Next we want to understand the regularity of $u_T$. If $K_X$ is big and nef ($T = \infty$) it follows from general geometric theory (see [TZ], [Z]), that there exists a holomorphic mapping $F : X \to \mathbb{P}^N$, (this is the associated map to $|K_X|$, recall that $F(X)$ may be singular) such that $\widehat{\omega}_\infty = F^*\omega_{FS}$ ($\omega_{FS}$ is the Fubini-Study form on $\mathbb{P}^N$). By Theorem 3.2.7 we get another corollary:

**Corollary 4.1.6** ([Z], [DZ])**.** *The potential $u_\infty$ is continuous on $X$ (and smooth outside some analytic set).*

In the case $T < \infty$ (i.e. if $K_X$ is only big) the continuity of the potential is still an open problem, since there is no correspondent theorem of existence of such associated map (see [Z]). In both cases, however, geometers expect the maximal possible regularity of $u_T$, that is $u_T \in \mathcal{C}^{1,1} \setminus \mathcal{C}^2$. Further analysis of singularities of $u_T$ (such as Hölder continuity) is crucial for better understanding of the geometry of the space $(X, \omega_T)$. If $T < \infty$ this is a (pseudo)metric space which is not complete, and geometers expect (see [ST1], [ST2]), that its completion $(\overline{X}, \overline{\omega}_T)$ should be some kind of bimeromorphic modification of $X$. There is a conjecture (supported by some 2-dimensional computations and examples - see, for example, [CN], [ST1]), that the Kähler-Ricci flow in the big $K_X$ case after finitely many such modifications would lead to a manifold (or an analytic space with mild singularities) $X_{can}$ on which $K_{X_{can}}$ is nef. Thus one would get a metric version of the famous minimal model program from algebraic geometry.

Finally we would like to refer to the papers [T], [TZ], [TZh], [ST1], [ST2], [CN], [To], [Ts], [EGZ] and [BEGZ], where one can find a much broader picture of the developed ideas, and also one can see the connections between geometry and pluripotential theory.

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