1. Where possible, find the upper and lower bounds of the following functions. Indicate which functions actually attain the bounds.

a) \( f(x) = \frac{1}{1 + x^2}; \quad D_f = \mathbb{R}. \)

b) \( f(x) = \frac{2}{e^x + e^{-x}}; \quad D_f = \mathbb{R}. \)

c) \( f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \quad D_f = \mathbb{R}. \)

d) \( f(x) = x\sqrt{1 - x^2}; \quad D_f = (-1, 1). \)

e) \( f(x) = 2 + x - x^2 \left( x - 1 \right)^2; \quad D_f = (1, +\infty). \)

f) \( f(x) = x^2 + x - 1 \); \( D_f = (1, +\infty). \)

g) \( f(x) = \frac{x^2 + x - 1}{x - 1}; \quad D_f = \mathbb{R} \setminus \{1\}. \)

h) \( f(x) = \cos \frac{1}{x}; \quad D_f = \mathbb{R} \setminus \{0\}. \)

i) \( f(x) = e^{-x} \cos x; \quad D_f = (0, +\infty). \)

j) \( f(x) = e^{-x} \cos x; \quad D_f = [0, +\infty). \)

2. Determine if the following functions have an inverse. In the cases that they do, give the domain of the inverse function and determine if it is continuous.

a) \( f(x) = x^2; \quad D_f = [-2, 1]. \)

b) \( f(x) = x^2; \quad D_f = [-2, -1]. \)

c) \( f(x) = x^3; \quad D_f = \mathbb{R}. \)

d) \( f(x) = 2x^3; \quad D_f = \mathbb{R}. \)

e) \( f(x) = \text{ctg} x; \quad D_f = (-2\pi, -\pi). \)

f) \( f(x) = \frac{ax + b}{cx + d}; \quad c \neq 0; \quad D_f = \mathbb{R} \setminus \{-\frac{d}{c}\}. \)

\( g) f(x) = \left( \frac{e^x - e^{-x}}{2} \right); \quad D_f = \mathbb{R}. \)

h) \( f(x) = \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right); \quad D_f = \mathbb{R} \setminus \{0\}. \)

3. a) Show that the function \( f(x) = \frac{1}{x} \) satisfies a Lipschitz condition on \((\varepsilon, +\infty), \varepsilon > 0,\) i.e., there is a positive constant \( L_\varepsilon(f) \) such

\[ |f(x) - f(y)| \leq L_\varepsilon(f)|x - y| \]

for all \( x, y \in (\varepsilon, +\infty). \) Conversely, show that no Lipschitz condition is possible for \( f \) on \((0, +\infty).\)

b) Show that \( f(x) = x^2 \) satisfies a Lipschitz condition on any bounded interval but does not on the open intervals \(( -\infty, 0), (0, +\infty) \) and \(( -\infty, +\infty)\)

c) Show that a Lipschitz function with bounded domain is a bounded function.
4. Give an example of a continuous function \( f(x) \), with domain \( D_f = [0, 1) \), such that the range = \( f(D_f) \) is the interval:
   a) \( (0, 1] \), b) \([0, 1] \), c) \((0, 1) \);
   d) \([0, +\infty) \), e) \((0, +\infty) \), f) \((−\infty, +\infty) \).

5. a) Explain why among the triangles inscribed in a circle of radius \( R \) there is one with maximal area.
   b) A mountain climber climbs a mountain with a stop watch timing the ascent. At the top, the climber immediately turns around, resets the timer to zero, and begins to descend along the same path. Show that there is a point on the path where the travel time up is the same as the travel time down.

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