Temat III

Induction, binomial Theorem, inequalities.

1. Show that for any \( n \geq 5 \) we have

\[ 2^n > n^2. \]

2. Show that for any \( n \in \mathbb{N} \setminus \{1\} \) we have \( 7|(n^7 - n) \).

3. Show that for any \( n \in \mathbb{N} \setminus \{1\} \) we have \( 42|(n^7 - n) \)

4. Show that for any \( n \in \mathbb{N} \) we have

\[
1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.
\]

5. Show that for any \( n \in \mathbb{N} \) we have

\[
1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

6. Find a general formula for the sum:

\[ 3 \cdot 7 + 7 \cdot 11 + 11 \cdot 15 + \cdots + (4n-1)(4n+3). \]

7. Show that for any \( n \in \mathbb{N} \) we have

\[
\frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} + \cdots + \frac{1}{(4n-1)(4n+3)} = \frac{n}{3(4n+3)}.
\]

8. Using the binomial theorem find a general formulas for:

\[
(1 + 1)^4 = \cdots \\
(2 + 1)^4 = \cdots \\
(3 + 1)^4 = \cdots \\
\vdots \\
(n + 1)^4 = \cdots
\]

and next, using the results of exercices 4 i 5 find a general patern for:

\[ 1^3 + 2^3 + 3^3 + \cdots + n^3. \]

9. Assuming that we already know the general paterns for the following sums:

\[
S^1(n) \overset{\text{def}}{=} 1^1 + 2^1 + 3^1 + \cdots + n^1 \\
S^2(n) \overset{\text{def}}{=} 1^2 + 2^2 + 3^2 + \cdots + n^2 \\
S^3(n) \overset{\text{def}}{=} 1^3 + 2^3 + 3^3 + \cdots + n^3 \\
\vdots \\
S^k(n) \overset{\text{def}}{=} 1^k + 2^k + 3^k + \cdots + n^k
\]

find a general formula for the sum:

\[ S^{k+1}(n) \overset{\text{def}}{=} 1^{k+1} + 2^{k+1} + 3^{k+1} + \cdots + n^{k+1}. \]
10. Show that for any \( n \in \mathbb{N} \) we have
\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n.
\]

11. Show that for any \( n \in \mathbb{N} \) we have
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.
\]

12. For any \( n \in \mathbb{N} \) calculate:
\[
-1 \left( \binom{n}{1} \right) + 2 \binom{n}{2} - 3 \binom{n}{3} + \cdots + (-1)^{n-1} (n-1) \binom{n}{n-1} + (-1)^n n \binom{n}{n}.
\]

13. Using the binomial theorem show that for any \( n \in \mathbb{N} \) the Bernoulli inequality with \( n \) as a power and \( x \geq 0 \), is true.

14. Using the binomial theorem show that for any \( n \in \mathbb{N} \) the Bernoulli inequality with \( n \) as a power and \( x > -1 \), is true.

*Hint: divide the interval \((-1, 0)\) into two subintervals and consider 2 cases: \( x \in (-1, a) \) and \( x \in [a, 0) \). How should you choose the value of \( a \) ?*

15. Show that for any nonnegative real numbers \( a, b \) we have
\[
\frac{a + b}{2} \geq \sqrt{ab}.
\]

16. Prove the inequality between an „arithmetic mean” and a „geometric mean” for any number \( n \) of nonnegative real numbers .

17. Show that for any \( x > 0 \) and \( y > 0 \) the inequality
\[
(x + y)^n \leq 2^{n-1} (x^n + y^n)
\]
holds true for any \( n \in \mathbb{N} \).

18. Show that for any \( n \in \mathbb{N} \) we have
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n + 1)} < 1.
\]

19. Show that for any \( n \in \mathbb{N} \) we have
\[
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2.
\]

20. Show that for any \( n \in \mathbb{N} \) we have
\[
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.
\]

21. Show that for any \( n \in \mathbb{N} \) we have
\[
\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n}}.
\]

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